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Scaling Limits in Quantum Models of Field-Particle Interaction

III. Taking a quantum detour

Effective Approximation and Dynamics of Many-Body Quantum Systems
Metz; June 2024.

Taking a Quantum Detour: General Scheme

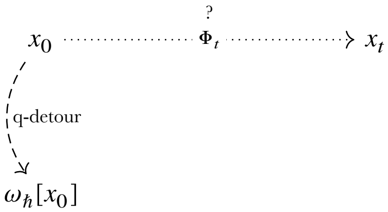
The idea: Quantum theory as a regularizer

- As it is apparent when considering the Hydrogen atom, quantization may act as a *regularizer* for classical singularities
- ⚠ New singularities may appear (*e.g.* van Hove I-infrared singularity) ⚠
- The idea of taking a quantum detour is to exploit quantum mechanics (and the *correspondence principle*) as a regularizer for classical problems
- Since the correspondence principle appears to be transparent to quantum generated singularities (QFT renormalization – see Lecture II), this approach seems reasonable/promising.

The scheme: Lifting – Evolving – Converging

$$x_0 \;\cdots\cdots\; \overset{?}{\Phi_t} \;\cdots\cdots\!\!\succ\; x_t$$

The scheme: Lifting – Evolving – Converging



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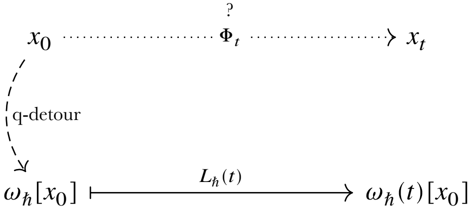

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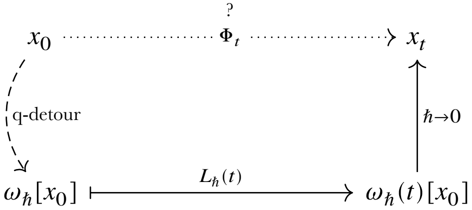

The precise scheme: $\hbar \to 0$ Liouville – Liouville↑Flow

$$\omega_\hbar(t)(\,\cdot\,) = \omega_\hbar(\,\cdot\,) - i\int_0^t \omega_\hbar(\tau)(\mathrm{ad}H_\hbar\,\cdot\,)\mathrm{d}\tau$$

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$$\Big\downarrow \; {\scriptstyle \omega_\hbar \to \mu,\ \hbar \to 0}$$

$$\exists (t \mapsto \mu_t) \text{ s.t. } \mu_t = \mu + \int_0^t \nabla^{\mathrm{t}}(X_{H_0}\mu_\tau)\mathrm{d}\tau$$

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$\forall x \exists \omega_\hbar[x] \to \delta_x$

$\omega_\hbar \to \mu$ $\hbar \to 0$

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μ-a.e. $\exists(!)\Phi_t$

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A related idea: quantum superselection of classical trajectories [Chabu 2016 (also Fermianian-P.Gérard-Lasser 2012)]

- Take a particle moving in the classical potential $V(q) = -|q|$. The classical dynamical theory lacks uniqueness: trajectories passing through $(q, p) = (0, 0)$ *bifurcate*.

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$$(q_0, p_0) \xrightarrow{\quad \Phi_t \quad} (0,0)$$

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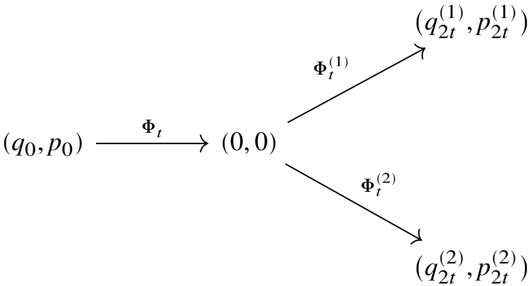
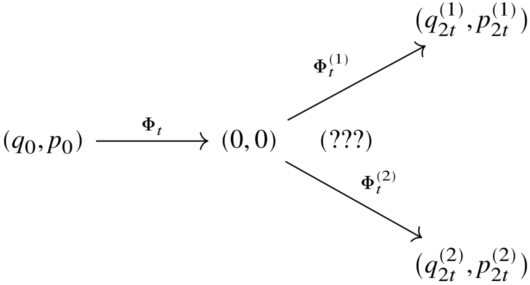


- C. Particle: «Which way to go?»



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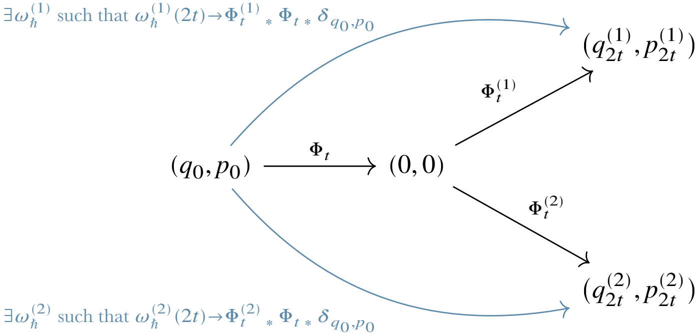

- C. Principle: «I'll guide you!»

Quantum detour: a proof of concept

Classical rigid charges interacting with the EM field

- Newton–Maxwell Equations:

$$
\text{(N-M)} \qquad \begin{cases}
\dot{\mathbf{q}}_j = \frac{\mathbf{p}_j}{m_j} \\
\dot{\mathbf{p}}_j = m_j(\varrho_j * \mathbf{E})(\mathbf{q}_j) + \mathbf{p}_j \times (\varrho_j * \mathbf{B})(\mathbf{q}_j) - \nabla_j V(\mathbf{q}) \\
\\
\partial_t \mathbf{B}(\cdot) + \nabla \times \mathbf{E}(\cdot) = 0 \\
\partial_t \mathbf{E}(\cdot) - \nabla \times \mathbf{B}(\cdot) = -\sum_j \frac{\mathbf{p}_j}{m_j} \varrho_j(\,\cdot - \mathbf{q}_j) \\
\\
\nabla \cdot \mathbf{E}(\cdot) = \sum_j \varrho_j(\,\cdot - \mathbf{q}_j) \\
\nabla \cdot \mathbf{B}(\cdot) = 0
\end{cases}
$$

Folklore: Disasters with (Almost) Point Charges

- Point Charges:

$$\varrho_j = e_j \delta \quad \Longrightarrow \quad \lightning$$

(electrostatic energy unbounded from below, atomic collapse by radiation)

- Charges with a small radius:

$$\varrho_j = e_j \mathbb{1}_{\left\{ |\cdot| < \frac{2e_j^2}{3m_j} \right\}} \quad \Longrightarrow \quad \lightning$$

(existence of runaway and non-causal solutions)

Classical Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}_{\mathrm{b}}^2$):

$$\varrho_j \text{ "regular enough"} \implies \text{GWP on suitable Sobolev spaces for } \mathbf{E} \text{ and } \mathbf{B}$$

$$\left(\varrho_j \in H^1 \implies \text{GWP for } \mathbf{E} \in \left(H^{\frac{1}{2}}\right)^{\times 3} \text{ and } \mathbf{B} \in \left(H^{\frac{1}{2}}\right)^{\times 3}\right)$$

Quantum rigid charges interacting with the EM field (Coulomb Gauge – Fock representation)

- Pauli-Fierz Hamiltonian: $L^2(\mathbb{R}^{3n}) \otimes \Gamma_s(L^2(\mathbb{R}^3, \mathbb{C}^2))$

$$\hat{H}_\hbar = \sum_{j=1}^{n} \frac{1}{2m_j} \left(\hat{p}_j - A_j(\hat{q}_j, \hat{a})\right)^2 + V(\hat{q}) + \hat{H}_\mathrm{f} ,$$

$$A_j(\hat{q}_j, \hat{a}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\epsilon_\lambda(k)}{\sqrt{2|k|}} \Big(\overline{\mathscr{F}\varrho_j}(k)\, \hat{a}_\lambda(k) e^{2\pi i k\cdot \hat{q}_j} + \mathscr{F}\varrho_j(k)\, \hat{a}^*_\lambda(k) e^{-2\pi i k\cdot \hat{q}_j} \Big) \mathrm{d}k ,$$

$$\hat{H}_\mathrm{f} = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} |k| \hat{a}^*_\lambda(k) \hat{a}_\lambda(k) \mathrm{d}k ,$$

$$[\hat{q}_j, \hat{p}_\ell] = i\hbar\delta_{j\ell} ,\ [\hat{a}_\lambda(k), \hat{a}^*_\mu(k')] = \hbar\delta_{\lambda\mu}\delta(k-k') .$$

- Quantum Dynamics:

$$\omega_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \omega_\hbar e^{i\frac{t}{\hbar}\hat{H}_\hbar} .$$

Quantum Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}^2_{\mathrm{b}}$):

$$\varrho_j \in \dot{H}^{-1} \cap \dot{H}^{\frac{1}{2}} \implies \hat{H}_\hbar \text{ is self-adjoint on } D(\hat{p}^2) \cap D(\hat{H}_{\mathrm{f}}) \ .$$

Remarks

- More singular Vs are allowed (*e.g.* Coulomb)
- Folklore is that point charges shall be admissible, however it is still mathematically an open problem (a renormalization is required)
- Atoms are stable, and no runaway or non-causal solutions are present

The detour theorem

Theorem ([Ammari-M.F.-Hiroshima 2022])

$$\varrho_j \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \implies \text{(N-M) } \textit{GWP for } \mathbf{E} \in \left(H^{\sigma}\right)^{\times 3} \textit{ and } \mathbf{B} \in \left(H^{\sigma}\right)^{\times 3}$$

$$\boxed{0 \leq \sigma \leq \tfrac{1}{2}}$$

Proof of the proof of concept

- *Step 0: (Abstract(Fock) Semiclassical Analysis)*
  - Existence of Wigner Measures: (sufficient conditions on $\omega_\hbar$)

$$\omega_\hbar \underset{\hbar\to 0}{\longrightarrow} \mathrm{d}\mu(x)$$

(here $x = (\mathbf{q}, \mathbf{p}, \mathbf{E}, \mathbf{B})$)

- *Step 1: (Correspondence principle)*
  - Liouville dynamics

$$\begin{array}{ccc} \omega_\hbar & \xmapsto{\;e^{-\frac{i}{\hbar}t\hat{H}_\hbar}\,(\,\cdot\,)\,e^{\frac{i}{\hbar}t\hat{H}_\hbar}\;} & \omega_\hbar(t) \\ {\scriptstyle \hbar\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \hbar\to 0} \\ \mu & \xmapsto[\;\mathscr{L}_t\,(\,\cdot\,)\;]{} & \mu_t \end{array}$$

- *Step 2: From Liouville to a.e. GWP*
  - A priori uniqueness

$$\varrho_j \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \implies \text{There exists at most one } H^\sigma\text{-solution of (N-M)}$$

  - *Liouville flow:* [Rouffort 2018]

$$[\textit{A priori}\,!] \wedge [\exists \mu_t = \mathscr{L}_t \mu] \implies \exists!\, x(t) \text{ sol. of (N-M) for } \mu\text{-a.a. } x_0$$

- *Step 3: Globalizing*
  - Coherent states concentrate on all points

$$\forall x_0\ \exists \omega_\hbar[x_0] \text{ (coherent state of minimal uncertainty): } \omega_\hbar[x_0] \underset{\hbar\to 0}{\longrightarrow} \mathrm{d}\delta_{x_0}\ .$$

Thank you for the attention (III)