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Scaling Limits in Quantum Models of Field-Particle Interaction

II. The correspondence principle

Effective Approximation and Dynamics of Many-Body Quantum Systems
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Bohr's correspondence principle in physics

The principle

Bohr's principle – M. Born's version [Bohr 1920, Born 1933].

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Mathematical physics interpretation – in view of lecture I.

Given a physical theory modeled by a concrete realization of $\mathbb{W}_{\hbar}(\mathscr{T}, \varsigma)$ (*e.g.*, $\mathscr{P}(\varphi)_2$, φ^4_3, Sine-Gordon, Nelson, Pauli-Fierz models; nonrelativistic quantum mechanics), one must recover the corresponding model of classical physics in the limit $\hbar \to 0$. (Recall that our $\hbar$ is a mathematical parameter measuring how much we are deforming the commutative theory to a noncommutative one)

- Quantization is *natural*, but how robust is it?
    - The Wigner measures result of Lecture I is very general, but it implies that states could lose mass in the limit; furthermore, the correspondence holds only for the expectation of "a few" observables (surely, not many important physical ones like the canonical observables, the energy, ...)
    - Does the correspondence principle holds for the dynamics of a model? For its (ground state) energy, bound states?
- Establishing the correspondence principle is an important "sanity check":
    - For (candidates of) quantum gravity, it is a crucial problem to establish the correspondence principle, and one of the main obstacles in making models such as *loop quantum gravity* acceptable [Giesel-Thiemann 06-08].
    - Even in more "orthodox" field theories, it is unclear whether after renormalization the classical limit still behaves as expected (QED $\underset{\hbar\to 0}{\longrightarrow}$ CED?).

The van Hove model – A perfect Playground

Original setting [van Hove/Myiatake 52, Derezinski 2003]

- Consider a scalar field generated by an immovable source, whose charge distribution is given by $\mathscr{F}^{-1}f$.
- The van Hove Hamiltonian in Fock space takes the form

$$\boxed{H_\hbar(\omega,f) = \mathrm{d}\Gamma_\hbar(\omega) + a_\hbar(f) + a^*_\hbar(f)}\ .$$

- For later convenience, for any $\alpha \in \mathbb{R}$, let us define

$$L^2_\alpha(\mathbb{R}^d) := L^2\big(\mathbb{R}^d, \omega^\alpha(k)\mathrm{d}k\big)\ .$$

- Observe that the Fock space is the GNS Hilbert space for the Fock vacuum state $\omega_{\Omega_\hbar} \in \mathrm{Reg}_\hbar\big(\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d), \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2\big)$, $\hat{\omega}_{\Omega_\hbar}(\eta) = e^{-\frac{\pi^2\hbar}{2}\|\eta\|_2^2}$. In this representation,

$$W_\hbar(f) = e^{i\pi\left(a^*_\hbar(f)+a_\hbar(f)\right)}\ .$$

Theorem (Derezinski 2003)

Let $H_\hbar(\omega, f)$ *be the van Hove Hamiltonian in Fock space. Then:*

- [Infrared regular case] $f \in L^2_{-1} \cap L^2_{-2} \iff H_\hbar(\omega, f)$ *is self-adjoint, bounded from below,* $\inf \sigma(H_\hbar) = -\|f\|^2_{L^2_{-1}}$, *and there exists a unique Fock ground state* $|\psi^{\mathrm{gs}}_\hbar\rangle\langle\psi^{\mathrm{gs}}_\hbar|$, *with*

$$\boxed{\psi^{\mathrm{gs}}_\hbar = W_\hbar\left(\frac{i}{\pi\hbar\omega} f\right)\Omega_\hbar}$$

- [Infrared singularity of type I] $f \in L^2_{-1} \setminus L^2_{-2} \iff H_\hbar(\omega, f)$ *is self-adjoint, bounded from below,* $\inf \sigma(H_\hbar) = -\|f\|^2_{L^2_{-1}}$, *and there exists no Fock ground state.*
- [Infrared singularity of type II] $f \in L^2 \setminus L^2_{-1} \implies H_\hbar(\omega, f)$ *is self-adjoint and unbounded from below.*

- In the infrared regular case, $W_\hbar(\frac{i}{\pi\hbar\omega}f)$ is a unitary operator in the Fock space (dressing transformation), and

$$W_\hbar(\tfrac{i}{\pi\hbar\omega}f)^* H_\hbar(\omega,f) W_\hbar(\tfrac{i}{\pi\hbar\omega}f) = \mathrm{d}\Gamma_\hbar(\omega) - \|f\|^2_{L^2_{-1}} \ .$$

- With an infrared singularity of type I, $H_\hbar(\omega,f)$ is bounded from below by KLMN theorem, and no unitary diagonalization is implementable.

- With an infrared singularity of type II, $H_\hbar(\omega,f)$ is self-adjoint by the Nelson commutator theorem.

Infrared singularity of type I – refined analysis

- The spectral properties of Hamiltonians describing fields interacting with particles (van Hove, spin-boson, Nelson, Pauli-Fierz,...) have been widely studied by the mathematical physics community [Ammari, Amour, Arai, Bach, Ballesteros, Betz, T. Chen, Deckert, Derezinski, Faupin, Fröhlich, C. Gérard, Griesemer, Hasler, Hinrichs, Hiroshima, Lieb, Lörinczi, Loss, Matte, Minlos, Møller, Pizzo, Siebert, I.M. Sigal, Spohn, Sasaki, ...].
- The absence of a Fock ground state for bounded from below massless field theories is sometimes called *infrared catastrophe*.
- Intuitively, it is due to the fact that the Fock space is ill-suited to describe systems with *truly many particles/excitations*, and massless models might have many many particles, even at low energies (with a mass gap, an excitation cannot have arbitrarily small energy).

- Fix ω, and fix $f \in L^2_{-1} \setminus L^2_{-2}$. Define the regularized van Hove model by

$$H_\hbar(\Lambda) = \mathrm{d}\Gamma_\hbar(\omega) + a^*_\hbar(f_\Lambda) + a(f_\Lambda)$$

with

$$f_\Lambda(k) = 1_{|\cdot| \geq \Lambda^{-1}}(k) f(k) \ .$$

Lemma ([Arai 2020])

Let $|\psi_\hbar^\Lambda\rangle\langle\psi_\hbar^\Lambda|$ be the ground state of the infrared regular $H_\hbar(\Lambda)$. Then:

- $\lim_{\Lambda\to\infty} \langle \psi_\hbar^\Lambda, \mathrm{d}\Gamma_\hbar(1)\psi_\hbar^\Lambda\rangle = \infty$
- $\mathrm{w-}\lim_{\Lambda\to\infty} \psi_\hbar^\Lambda = 0$

Remark

- The physical interpretation of the above lemma is the following. The ground state $\psi_\hbar^\infty$ in the infrared singular-I case would be a Fock vector with *infinitely many field's excitations*: the so-called *soft photons*, infinitely many excitations with a very small energy (making the total energy bounded from below). However, by construction Fock vectors have smaller and smaller probability of having more and more particles, therefore $\psi_\hbar^\infty$ has zero amplitude of transition to any Fock vector.

The classical van Hove model

- $E(z) = \langle z, \omega z\rangle_2 + 2\mathrm{Re}\langle f, z\rangle_2$
- In the form above, the natural domain of definition of E seems to be $z \in L^2 \cap L^2_1$ (with $f \in L^2$, however E is bounded from below only if $f \in L^2_{-1}$). Therefore, it is convenient to rewrite E as an explicitly bounded from below functional

$$\boxed{E(z) = \langle z, \omega z\rangle_2 + 2\mathrm{Re}\langle f/\sqrt{\omega}, \sqrt{\omega} z\rangle_2 \Big(= \|z + f/\omega\|^2_{L^2_1} - \|f\|^2_{L^2_{-1}} \Big)}$$

with $f \in L^2_{-1}$ and domain of definition L^2_1.

Lemma

E is bounded from below by $-\|f\|^2_{L^2_{-1}}$, and it has a unique minimizer

$$\boxed{z^{\mathrm{gs}} = -\frac{f}{\omega}}$$

z^{gs} belongs to L^2_1 for any $f \in L^2_{-1}$.

Remarks

- The infrared singularity of type II is *both classical and quantum*.
- The infrared singularity of type I is *only quantum*! (typical in renormalization)
- The classical limit $\hbar \to 0$ shall be "transparent" to the I-infrared singularity.

The correspondence principle for the van Hove model (using abstract semiclassical analysis)

- We make the following assumption: $\boxed{L^2_{-1} \subset \mathscr{S}'}$. Also, let us define $\mathscr{S}_{\mathbb{R}}$ to be the usual Schwartz space seen as a real vector space. Let us remark that $\mathrm{Re}\langle \cdot , \cdot \rangle_2$ makes $\mathscr{S}_{\mathbb{R}}$ a real pre-Hilbert space, and $\mathrm{Im}\langle \cdot , \cdot \rangle_2$ a real symplectic space. Finally, let us define $\mathscr{S}'_{\mathbb{R}}$ to be the continuous dual of $\mathscr{S}_{\mathbb{R}}$ by means of the duality bracket $\mathrm{Re}\langle \cdot , \cdot \rangle_2$.

- Let Φ_t be the Hamiltonian flow on $\mathscr{S}'$ associated to the Hamiltonian E. The Hamilton equation of E reads

$$i\partial_t z = \omega z + f \; ,$$

whose solution for an initial datum z_0 is

$$\boxed{z(t) = e^{-it\omega}\Big(z_0 + \frac{f}{\omega}\Big) - \frac{f}{\omega}}$$

- The map Φ_t can be split in three maps on $\mathscr{S}'$: $\Phi_t = \tau_{\frac{f}{\omega}}^{-1} \circ \Phi_t^0 \circ \tau_{\frac{f}{\omega}}$, where
  - $\tau_{\frac{f}{\omega}}$ is the "phase space translation" by $\frac{f}{\omega} \in \mathscr{S}'$;
  - $\Phi_t^0 \in \mathscr{L}(\mathscr{S}')$ is a linear transformation $z \mapsto e^{-it\omega} z$ that preserves $\langle \cdot , \cdot \rangle_2$, and therefore it defines by transposition a linear symplectic map $\vartheta_{\Phi_t^0} : (\mathscr{S}_{\mathbb{R}}, \mathrm{Im}\langle \cdot , \cdot \rangle_2) \to (\mathscr{S}_{\mathbb{R}}, \mathrm{Im}\langle \cdot , \cdot \rangle_2)$, acting as $\eta \mapsto e^{it\omega}\eta$.
- A symplectic map on test functions, such as $\vartheta_{\Phi_t^0}$ induces a *-homomorphism $\mathbb{W}_\hbar(\vartheta_{\Phi_t^0})$ on $\mathbb{W}_\hbar(\mathscr{S}_{\mathbb{R}}, \mathrm{Im}\langle \cdot , \cdot \rangle_2)$, agreeing with the natural quantization $\mathfrak{q}_\hbar$:

$$\mathbb{W}_\hbar(\vartheta_{\Phi_t^0})[W_\hbar(\eta)] = W_\hbar(e^{it\omega}\eta)\Big(\text{“}=\text{”} e^{i\frac{t}{\hbar}\mathrm{d}\Gamma_\hbar(\omega)} W_\hbar(\eta) e^{-i\frac{t}{\hbar}\mathrm{d}\Gamma_\hbar(\omega)}\Big)$$

- On the other hand, it is well known that *quantum phase space translations are (uniquely) implemented by (suitably scaled) Weyl operators*. It is thus natural to define, by slight abuse of notation,

$$\mathfrak{q}_\hbar(\tau_{\frac{f}{\omega}})[W_\hbar(\eta)] = W_\hbar(\eta) e^{2\pi i \mathrm{Re}\left\langle \frac{f}{\omega}, \eta \right\rangle_2}\Big(\text{“}=\text{”} W_\hbar\big(\tfrac{1}{i\pi\hbar\omega} f\big)^* W_\hbar(\eta) W_\hbar\big(\tfrac{1}{i\pi\hbar\omega} f\big)\Big)$$

- To sum up, we can define a van Hove dynamical map $L_\hbar(t)$ on $\mathbb{W}_\hbar(\mathscr{S}_\mathbb{R}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2)$:

$$\boxed{L_\hbar(t)[A] = \mathbb{q}_\hbar(\tau_{-\frac{f}{\omega}}) \circ \mathbb{W}_\hbar(\vartheta_{\Phi_t^0}) \circ \mathbb{q}_\hbar(\tau_{\frac{f}{\omega}})[A]}$$

$$\left(\text{“}=\text{”}\; e^{i\frac{t}{\hbar}H_\hbar(\omega,f)}\, A\, e^{-i\frac{t}{\hbar}H_\hbar(\omega,f)}\right)$$

- $L_\hbar(t)$ induces by transposition a map on $\omega_\hbar \in \mathrm{Reg}_\hbar(\mathscr{S}_\mathbb{R}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2)$, with a very explicit action on the Fourier transform:

$$\boxed{\left(\widehat{L_\hbar(t)\omega_\hbar}\right)(\eta) = \hat{\omega}_\hbar(e^{it\omega}\eta)e^{2\pi i\mathrm{Re}\left\langle \frac{f}{\omega},(e^{it\omega}-1)\eta\right\rangle_2}}$$

Egorov Theorem – Propagation of chaos

Theorem 1 (Egorov-type theorem)

Let $\omega_\hbar \in \mathrm{Reg}_\hbar(\mathscr{S}_{\mathbb{R}}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2)$ *such that* $\exists \hbar_n \to 0$ *and* $\mathfrak{m} \in \mathscr{P}_{\mathrm{cyl}}(\mathscr{S}'_{\mathbb{R}}, \mathscr{S}_{\mathbb{R}})$ *such that* $\hat{\omega}_{\hbar_n} \underset{n\to\infty}{\longrightarrow} \hat{\mathfrak{m}}$. *Then* $\forall t \in \mathbb{R}$,

$$\boxed{\lim_{n\to\infty} \widehat{\left(L_{\hbar_n}(t)\,\omega_{\hbar_n}\right)} = \widehat{\left(\Phi_{t\,*}\,\mathfrak{m}\right)}}$$

In other words, the following "diagram" is commutative:

$$\begin{array}{ccc}
\omega_{\hbar_n} & \xrightarrow{L_{\hbar_n}(t)} & \omega_{\hbar_n}(t) \\
{\scriptstyle \hbar_n\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \hbar_n\to 0} \\
\mathfrak{m} & \xrightarrow[\Phi_t]{} & \Phi_{t\,*}\,\mathfrak{m}
\end{array}$$

Proof

- By $\hat{\omega}_{\hbar_n} \to \hat{\mathfrak{m}}$, we get

$$\lim_{n\to\infty} \widehat{\left(L_{\hbar_n}(t)\omega_{\hbar_n}\right)}(\eta) = \lim_{n\to\infty} \hat{\omega}_{\hbar_n}(e^{it\omega}\eta)e^{2\pi i \mathrm{Re}\left\langle \frac{f}{\omega},(e^{it\omega}-1)\eta\right\rangle_2}$$
$$= \hat{m}(e^{it\omega}\eta)e^{2\pi i \mathrm{Re}\left\langle \frac{f}{\omega},(e^{it\omega}-1)\eta\right\rangle_2} = \int_{\mathscr{S}'}^{\bullet} e^{2\pi i \mathrm{Re}\langle z(t),\eta\rangle_2} \mathrm{d}\mathfrak{m}(z) = \widehat{\left(\Phi_{t\,*}\,\mathfrak{m}\right)}(\eta)$$

□

Ground state energy and ground states

- The van Hove (W*-)dynamical system $t \mapsto L_\hbar(t)$ has a *generator* ("ad $H_\hbar(\omega, f)$") that can be abstractly defined, as well as its properties like the spectrum, ground states, KMS (equilibrium) states, …

- These of course agree with the concrete van Hove model we defined above in Fock space. In particular, the ground state energy $E_\hbar$ of the van Hove dynamical system is given by

$$E_\hbar = -\|f\|^2_{L^2_{-1}} \; .$$

- The ground state of this dynamical system is unique, and is given by the regular state $\omega_\hbar^{\mathrm{gs}} \in \mathrm{Reg}_\hbar(\mathscr{S}_\mathbb{R}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2)$ with Fourier transform

$$\boxed{\hat{\omega}_\hbar^{\mathrm{gs}}(\eta) = e^{-\frac{\pi^2\hbar}{2}\|\eta\|_2^2}\; e^{2\pi i \mathrm{Re}\left\langle -\frac{f}{\omega}, \eta\right\rangle_2}}$$

- ⚠ The algebraic ground state is defined for all sources f and dispersion relations ω such that $\frac{f}{\omega} \in \mathscr{S}'(\mathbb{R}^d)$. In particular, for the I-infrared singular van Hove model! ⚠
- In the I-infrared singular case, the GNS representation of $\omega_\hbar^{\mathrm{gs}}$ *is non-Fock* (it is inequivalent to the Fock representation), however it can be explicitly embedded in a Fock space [see Arai 2020].

  The idea is that in the non-Fock representation, the vacuum vector $\Omega_\hbar$ corresponds to $\omega_\hbar^{\mathrm{gs}}$ and the creation/annihilation/number operators create/annihilate/count only the non-soft excitations we build on top of the ground state.

Theorem 2 (Semiclassical GSE and GS)

Let $L_\hbar(t)$ be the van Hove dynamical system. Then:

- $\boxed{\lim_{\hbar\to 0} E_\hbar = E_0 = E(z_{\mathrm{gs}}) = -\|f\|^2_{L^2_{-1}}}$
- $\boxed{\lim_{\hbar\to 0} \omega_\hbar^{\mathrm{gs}} = \delta_{z_{\mathrm{gs}}}}$

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- $\boxed{\lim_{\hbar\to 0} \omega_\hbar^{\mathrm{gs}} = \delta_{z_{\mathrm{gs}}}}$

Proof.

- Step one is very difficult so let me skip the proof;
- Almost as difficult is to prove that

$$\lim_{\hbar\to 0} \hat{\omega}_\hbar^{\mathrm{gs}}(\eta) = \lim_{\hbar\to 0} e^{-\frac{\pi^2\hbar}{2}\|\eta\|_2^2}\, e^{2\pi i \mathrm{Re}\left\langle -\frac{f}{\omega},\eta\right\rangle_2} = e^{2\pi i \mathrm{Re}\left\langle -\frac{f}{\omega},\eta\right\rangle_2}$$
$$= \int^{\bullet}_{\mathscr{S}'} e^{2\pi i\langle z,\eta\rangle_2} \mathrm{d}\,\delta_{z_{\mathrm{gs}}}(z) = \int_{\mathscr{S}'} e^{2\pi i\langle z,\eta\rangle_2} \mathrm{d}\,\delta_{z_{\mathrm{gs}}}(z) = \hat{\delta}_{z_{\mathrm{gs}}}(\eta)$$

□

The quantization is natural, but is it robust?

- For the van Hove model, quantization is *very robust*, in fact despite the possible appearance of quantum infrared singularities, the correspondence principle holds – irrespective of them – for both the dynamics and the ground state properties.

Not so perfect playgrounds – some m(M)L of sweat required

More realistic models of particle-field interaction, different scalings

- "Pure" semiclassical field theories ($\hbar \to 0$):
  - Schrödinger (Coherent states – Theorem 1) [Hepp 1974, Ginibre-Velo 1979]
  - $P(\varphi)_2$ model (Schwinger functions in the limit $\hbar \to 0$) [Eckmann 1977]
  - Schrödinger (Wigner measures – Theorem 1) [Ammari-Breteaux-M.F.-Liard-Nier-Pawilowski-Rouffort 2008-19]
  - $P(\varphi)_2$ model (Coherent states – Theorem 1) [Ammari-Zerzeri 2012]
  - Schrödinger (de Finetti measures – Theorem 1&2) [Lewin-Nam-Rougerie 2014-21]
  - Bose-Hubbard model on a graph (KMS states – Theorem 2.5) [Ammari-Farhat-Petrat-Ratsimanetrimanana 2021-24]
  - van Hove model (Wigner measures – Theorem 1&2) [M.F.-Fratini upcoming]

- Systems of many bosons ($N \sim \hbar^{-1}$) coupled with a semiclassical field ($\hbar \to 0$):
    - Nelson model with cutoff (Coherent states – Theorem 1) [M.F. 2013]
    - Nelson model with and without cutoff (Wigner measures – Theorem 1&2) [Ammari-M.F. 2014-17]
    - Nelson model with and without cutoff (α-method/Bogoliubov+Coherent states – Theorem 1) [M.F.-Lampart-Leopold-Mitrouskas-Petrat 2019-23]
    - Pauli-Fierz model (α-method+Coherent states – Theorem 1) [M.F.-Leopold-Pickl 2020-23]
    - Scattering for the Nelson model (Wigner measures – Theorem 1.5) [Ammari-M.F.-Olivieri 2023]
- Systems of semiclassical particles $\hbar \to 0$ coupled with a semiclassical field ($\hbar \to 0$):
    - Pauli-Fierz model (Wigner measures – Theorem 1) [Ammari-M.F.-Hiroshima 2022]
    - Nelson model (Wigner measures – Theorem 1) [Farhat 2024]

- Bipartite systems only a part semiclassical ("Quasi-classical limits", $\hbar \to 0$):
  - Nelson and Pauli-Fierz models, with and without cutoff (Wigner measures – Theorem $\sum_{n=0}^{100} 2^{-n}$) [Breteaux-Correggi-M.F.-Olivieri-Faupin 2018-24]
  - Spin-Boson, Pauli-Fierz models (different take on semiclassical analysis – Theorem 1+) [Amour-Jager-Khodja-Lascar-Nourrigat 2013-2020]
  - Nelson, Polaron, Pauli-Fierz models (Wigner measures – Theorem 1&2) [Correggi-M.F.-Olivieri 2023]
  - Nelson, Polaron (Wigner measures & Concentration compactness – Theorem 2) [M.F.-Olgiati-Rougerie 2023]
  - Polaron with point interaction (Coherent states – Theorem 1) [Carlone-Correggi-M.F.-Olivieri 2021]
  - Spin-Boson model (Wigner measures – Theorem 1 and decoherence) [Correggi-M.F.-Fantechi-Merkli 2023-24]
  - Caldeira-Leggett model (Wigner measures – Theorem 1&2) [Correggi-M.F.-Fantechi upcoming]

Why all the sweat?

- The quantum and classical evolutions for the models above ($\setminus${van Hove , spin-boson}) are not trivial nor explicit, and their generators are not diagonalizable, the quantum ground state energy depends on $\hbar$.
- The (Ammari-Nier) strategy for proving Theorem 1 is the following:
  - Write the evolution of the expectation of the Weyl operator as an integral (Duhamel) equation;
  - Take the limit $\hbar \to 0$ of such equation, to obtain a classical transport (Liouville) equation for the Wigner measures;
  - In order to do that, one shall prove that it is possible to extract a common subsequence $\hbar_n \to 0$ for convergence of $\omega_\hbar(t)$ to μ_t at all times (using uniform number operator/Hamiltonian bounds at all times);
  - One studies the uniqueness properties of the classical Liouville equation, under the *a priori* regularity properties of the Wigner measure evolution $t \mapsto \mu_t$.

- The (Ammari) strategy for proving Theorem 2 is the following:
  - Energy upper bound (easy): coherent trial states (states of minimal uncertainty – Wigner measures are deltas!);
  - Energy lower bound: take a minimizing sequence $\psi_\hbar$, and take the liminf of

$$\langle \psi_\hbar, H_\hbar \psi_\hbar \rangle < E_\hbar + o_\hbar(1) \; ;$$

  - The convergence of the ground state uses the same strategy as the energy lower bound.

Thank you for the attention (II)