Marco Falconi

Politecnico di Milano

Scaling Limits in Quantum Models of Field-Particle Interaction

I. Abstract Semiclassical Analysis

Effective Approximation and Dynamics of Many-Body Quantum Systems
Metz; June 2024.

What is (abstract) Quantization?

"First quantization is a mystery, but second quantization is a functor"
(E. Nelson)

Fields are Rough

In field theories, it is often necessary – or natural – to consider canonical observables (the fields themselves) that are not smooth:

- The energy of the electromagnetic field is given by

$$\int_{\Omega} \Big(\mathbf{E}^2(x) + \mathbf{B}^2(x) \Big) \mathrm{d}x \ ,$$

  a quantity naturally related to the Lebesgue space $L^2(\Omega, \mathrm{d}x) \subset \mathscr{S}'(\Omega)$.

- A local quantum (scalar) field $\hat{\varphi}(x)$ makes sense (mathematically) only as a so-called *operator-valued distribution*.

Test Functions

To study rough functions, we need to test their action on smooth functions:

- Let us denote by $\mathscr{T}$ the (real) vector space of *test functions*.
- As in finite dimensional classical mechanics, we shall endow the space of fields with a symplectic structure.

  ⚠ Fields are rough ⚠

  The symplectic structure is given to test functions only (as in classical mechanics, it involves the multiplication of canonical observables).
- Let us denote by $\varsigma : \mathscr{T} \times \mathscr{T} \to \mathbb{R}$ a non-degenerate, bilinear, antisymmetric form on $\mathscr{T}$.
- $(\mathscr{T}, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$ is the symplectic space of test functions

The functor of classical observables

- Given the space of test functions $\mathscr{T}$, let $\mathscr{T}^\star$ be your favorite (and "big enough") dual space of (classical) fields.
  - If $\mathscr{T}$ has no additional structure, the natural choice is $\mathscr{T}^\star = \mathscr{T}^*$ the algebraic dual.
  - If $\mathscr{T} \in \mathbf{TVS}_{\mathbb{R}}$ is a topological vector space, another natural choice is $\mathscr{T}^\star = \mathscr{T}'$ the continuous dual (the space of distributions).
- For all our purposes, this freedom in the choice of the space of fields $\mathscr{T}^\star$ is *fictitious*: the choice of test functions $\mathscr{T}$ unambiguously determines the physical output of our semiclassical theory.
- $\mathscr{T}$ determines a set of *Fourier characters* $\{\gamma_f\}_{f\in\mathscr{T}}$ on $\mathscr{T}^\star$:

$$\begin{aligned} \gamma_f : \mathscr{T}^\star &\longrightarrow U(1) \\ T &\longmapsto e^{2\pi i \langle T, f\rangle} \end{aligned}$$

- From characters, one can build *trigonometric polynomials*:

$$\Pi_n(T) = \sum_{j=1}^{n} \lambda_j \gamma_{f_j}(T) \ .$$

- The closure of trigonometric polynomials with respect to the uniform norm

$$\|\Pi_n\|_\infty := \sup_{T \in \mathscr{T}^\star} |\Pi_n(T)|$$

is called the space of *Almost Periodic functions* $\mathbb{AP}(\mathscr{T})$ on $\mathscr{T}^\star$ generated by $\mathscr{T}$.
With the uniform norm and the pointwise product, $\mathbb{AP}(\mathscr{T})$ is an abelian unital C*-algebra ($\gamma_0 = \mathrm{id}$), generated by the characters:

$$\mathbb{AP}(\mathscr{T}) = \mathrm{C}^*\left\{\gamma_f \ , f \in \mathscr{T}\right\} .$$

Definition (The functor of classical observables $\mathbb{W}_0$)

The functor of classical observables $\mathbb{W}_0 : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{C^*alg}$ is defined by

$$\mathbb{W}_0(\mathscr{T}, \varsigma) = \mathbb{AP}(\mathscr{T}) \ ,$$

and given a symplectic linear morphism $\vartheta : (\mathscr{T}, \varsigma) \to (\mathscr{U}, \zeta)$,

$$\mathbb{W}_0(\vartheta)[\gamma_f] = \gamma_{\vartheta f}$$

that extends to a *-homomorphism by linearity.

Remarks

- These observables are not "many" (the physical energies, or canonical variables themselves are not included), but enough to make a meaningful quantization (and semiclassical analysis).

- The symplectic form does not play a role in defining classical observables (also in standard classical mechanics, it comes into play when one looks at the dynamics), however it is necessary to define quantum observables that are consistent with the classical limit.

The functor of quantum observables

- Classical observables form an abelian C*-algebra, while quantum observables shall form a non-abelian C*-algebra (that by the GNS construction is always concretely represented as an algebra of operators in a Hilbert space).
- Let us *deform* the product between abelian characters γ_f, i.e.

$$\gamma_f \gamma_g = \gamma_{f+g} \ ,$$

to a non-abelian one. How shall we do it? Let us introduce a *semiclassical parameter $\hbar$* such that the quantum Fourier characters $\{W_\hbar(f)\}_{f\in\mathscr{T}}$ satisfy the composition rule

$$W_\hbar(f)W_\hbar(g) = W_\hbar(f+g)\alpha_\hbar(f,g) \ ,$$

for some suitable $\alpha_\hbar(f,g)$. Rules for $\alpha_\hbar$:

  - $\alpha_\hbar(f,g) \neq \alpha_\hbar(g,f)$ (non-abelian product)
  - $\alpha_\hbar(f,g) = 1 + o_\hbar(1)$ ($\hbar$-deformation)

- P.A.M. Dirac: «A Quantum commutator is approximated, at first order in $i\hbar$, by the corresponding classical Poisson bracket»

  H. Weyl: «Then we shall choose $\alpha_{\hbar}(f, g) = e^{-i\pi^2\hbar\varsigma(f,g)}$»

- The *Weyl operators* (noncommutative characters of quantum mechanics) are thus defined as the collection $\{W_{\hbar}(f)\}_{f\in\mathscr{T}}$ satisfying:
  - $W_{\hbar}(f) \neq 0$
  - $W_{\hbar}(f)^* = W_{\hbar}(-f)$
  - $W_{\hbar}(f)W_{\hbar}(g) = W_{\hbar}(f+g)e^{-i\pi^2\hbar\varsigma(f,g)}$

- From their definition, it follows that $W_{\hbar}(0) = \mathrm{id}$, $W_{\hbar}(f)^* = W_{\hbar}(f)^{-1}$, $\|W_{\hbar}(f)\| = 1$.
- Let us define the Weyl algebra of Canonical Commutation Relations (CCR-algebra)

$$\mathbb{CCR}_{\hbar}(\mathscr{T}, \varsigma) = \mathrm{C}^*\Big\{W_{\hbar}(f)\ ,\ f \in \mathscr{T}\Big\}\ .$$

Definition (The functor of quantum observables $\mathbb{W}_\hbar$ [I.E. Segal 59-61])

The functor of quantum observables $\mathbb{W}_\hbar : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{C^*alg}$ is defined by

$$\mathbb{W}_\hbar(\mathscr{T}, \varsigma) = \mathbb{CCR}_\hbar(\mathscr{T}, \varsigma) \ ,$$

and given a symplectic linear morphism $\vartheta : (\mathscr{T}, \varsigma) \to (\mathscr{U}, \zeta)$,

$$\mathbb{W}_\hbar(\vartheta)[W_\hbar(f)] = W_\hbar(\vartheta f)$$

that extends to a *-homomorphism by linearity.

Quantization

- Let us define a *natural transformation* $\mathbb{q}_\hbar : \mathbb{W}_0 \to \mathbb{W}_\hbar$ as follows:
  - If $(\mathscr{T}, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, the component $\mathbb{q}_\hbar^{\mathscr{T}} : \mathbb{W}_0(\mathscr{T}, \varsigma) \to \mathbb{W}_\hbar(\mathscr{T}, \varsigma)$ of the natural transformation is defined by its action on trigonometric polynomials:

$$\mathbb{q}_\hbar^{\mathscr{T}}\left(\sum_j \lambda_j \gamma_{f_j}\right) = \sum_j \lambda_j W_\hbar(f_j) \ .$$

  - Given $\vartheta : (\mathscr{T}, \varsigma) \to (\mathscr{U}, \zeta)$, the diagram

$$\begin{array}{ccc} \mathbb{W}_0(\mathscr{T}, \varsigma) & \xrightarrow{\mathbb{q}_\hbar^{\mathscr{T}}} & \mathbb{W}_\hbar(\mathscr{T}, \varsigma) \\ {\scriptstyle \mathbb{W}_0(\vartheta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{W}_\hbar(\vartheta)} \\ \mathbb{W}_0(\mathscr{U}, \zeta) & \xrightarrow[\mathbb{q}_\hbar^{\mathscr{U}}]{} & \mathbb{W}_\hbar(\mathscr{U}, \zeta) \end{array}$$

  is clearly commutative, so $\mathbb{q}_\hbar$ is a natural transformation indeed.

Definition (Abstract Weyl quantization)

Given an almost periodic classical field observable $a : \mathscr{T}^\star \to \mathbb{C}$, its *abstract Weyl quantization* is given by $\mathfrak{q}_\hbar^{\mathscr{T}}(a)$.

Remarks

- The rule of (abstract) Weyl quantization is very simple: substitute each $\mathscr{T}$-Fourier character with its quantum deformation, *i.e.* a Weyl operator.
- Quantum mechanical case: $\boxed{\mathscr{T} = \mathscr{T}^{\star} = \mathbb{R}^{2d}}$ (with the canonical symplectic form).
  - Any function $a \in \mathscr{F}^{-1}L^1(\mathbb{R}^{2d}, \mathrm{d}x) \subset C_0(\mathbb{R}^{2d})$ also belongs to $\mathbb{AP}(\mathbb{R}^{2d})$. The function $\alpha_k(\cdot) = \hat{a}(k)e^{2\pi ik\cdot}$ is Bochner integrable with respect to dk:

    $$\|\alpha_k\|_\infty = \sup_{x\in\mathbb{R}^{2d}} |\alpha_k(x)| = |\hat{a}(k)| \ , \ \int_{\mathbb{R}^{2d}} \|\alpha_k\|_\infty \mathrm{d}k = \|\hat{a}\|_1 \ ;$$

    therefore, there exist a Cauchy sequence – in the $L^\infty(\mathbb{R}^{2d}, \mathrm{d}x)$ norm – of trigonometric polynomials

    $$s_n(\cdot) = \sum_j \lambda(E_j)\hat{a}(k_j)\gamma_{k_j}(\cdot)$$

    such that $\lim_{n\to\infty} s_n(\cdot) = a(\cdot)$, thus proving that $a \in \mathbb{AP}(\mathbb{R}^{2d})$.
  - $\boxed{\mathbb{q}_\hbar^{\mathbb{R}^{2d}}(a) = \int_{\mathbb{R}^{2d}} \hat{a}(k)W_\hbar(k)\mathrm{d}k}$ – the *standard Weyl quantization formula* (for symbols in $\mathscr{F}^{-1}L^1$).

An answer to Ed

- Ed:
  «First quantization is a mystery, but second quantization is a functor»

An answer to Ed

- Me:

  «Second quantization is a functor, but quantization is a natural transformation»

Abstract Semiclassical Analysis

Classical states

- Let us denote the cone of all algebraic states on $\mathbb{W}_0(\mathscr{T}, \varsigma)$ by $\mathbb{W}_0(\mathscr{T}, \varsigma)'_{+,1}$.
- The *regular states* $\mathrm{Reg}_0(\mathscr{T}, \varsigma) \subset \mathbb{W}_0(\mathscr{T}, \varsigma)'_{+,1}$ are defined as follows:

$$\omega_0 \in \mathrm{Reg}_0(\mathscr{T}, \varsigma) \ \text{ iff } \ \forall f \in \mathscr{T} \ , \ \lambda \mapsto \omega_0(\gamma_{\lambda f}) \text{ is a continuous map}$$

Lemma

There is a bijection between regular states $\mathrm{Reg}_0(\mathscr{T}, \varsigma)$ *and cylindrical probabilities* $\mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$.

What is a cylindrical probability?

- The *σ-algebra of cylinders* $\Sigma(\mathscr{T}^\star, \mathscr{T})$ is the initial σ-algebra, *i.e.* the smallest σ-algebra that makes all elements of $\mathscr{T}$ (seen as functions on $\mathscr{T}^*$) measurable. (In many concrete examples, *e.g.* if $\mathscr{T}^*$ is a separable Banach space, the σ-algebra of cylinders coincides with the Borel σ-algebra)
- A *cylindrical probability* $\mathfrak{m} \in \mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ is a finitely additive measure on $\Sigma(\mathscr{T}^\star, \mathscr{T})$ with $\mathfrak{m}(\mathscr{T}^\star) = 1$, that is σ-additive when restricted to $\Sigma(\mathscr{T}^*, F)$ with F any finite dimensional subspace of $\mathscr{T}$.

Theorem (Bochner)

The Fourier transform is a bijection between $\mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ and the functions $\mathscr{G} : \mathscr{T} \to \mathbb{C}$ satisfying the following properties:

- $\mathscr{G}(0) = 1$;
- $\sum_{j,k} \bar{\alpha}_k \alpha_j \mathscr{G}(f_j - f_k) \geq 0$;
- $\mathscr{G}\big|_F$ *is continuous for all F finite dimensional subspace of $\mathscr{T}$.*

Proof of the Lemma

- Let $\omega_0 \in \mathrm{Reg}_0(\mathscr{T}, \varsigma)$. Then $\mathscr{G}_{\omega_0}(f) := \omega_0(\gamma_f)$ satisfies:
  - $\mathscr{G}_{\omega_0}(0) = 1$;
  - $\sum_{j,k} \bar{\alpha}_k \alpha_j \mathscr{G}_{\omega_0}(f_j - f_k) = \omega_0\Big(\big(\sum_k \alpha_k \gamma_{f_k}\big)^* \big(\sum_j \alpha_j \gamma_{f_j}\big)\Big) \geq 0$;
  - $\mathscr{G}_{\omega_0}\big|_F$ is continuous for all finite dimensional $F \subset \mathscr{T}$ (thanks to the regularity of ω_0).

  Then by Bochner theorem $\mathrm{Reg}_0(\mathscr{T}, \varsigma)$ injects into $\mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$.
- Now, let us consider $\mathfrak{m} \in \mathscr{P}_{\mathrm{cyl}}$. Any trigonometric polynomial $\Pi_n(T) = \sum_{j=1}^n \lambda_j e^{2\pi i \langle T, f_j \rangle}$ is a $\Sigma(\mathscr{T}^\star, \mathrm{span}\{f_1, \dots, f_n\})$-measurable function, and since $\mathfrak{m}$ restricted to $\Sigma(\mathscr{T}^\star, \mathrm{span}\{f_1, \dots, f_n\})$ is a probability measure, we can define the functional

$$\omega_{0,\mathfrak{m}}(\Pi_n) = \int_{\mathscr{T}^\star}^{\bullet} \Pi_n(T) \mathrm{d}\mathfrak{m}(T) \left(= \sum_{j=1}^n \lambda_j \hat{\mathfrak{m}}(f_j) \right);$$

  and easily check that it is positive, linear and bounded, with $\omega_{0,\mathfrak{m}}(\mathrm{id}) = 1$. Therefore, since trigonometric polynomials are dense in $\mathbb{W}_0(\mathscr{T}, \varsigma)$, $\omega_{0,\mathfrak{m}}$ extends uniquely to a regular state (regularity follows from the continuity of $\hat{\mathfrak{m}}$).

□

Quantum states

- Let us denote the cone of all algebraic states on $\mathbb{W}_\hbar(\mathscr{T},\varsigma)$ by $\mathbb{W}_\hbar(\mathscr{T},\varsigma)'_{+,1}$.
- The *regular states* $\mathrm{Reg}_\hbar(\mathscr{T},\varsigma) \subset \mathbb{W}_\hbar(\mathscr{T},\varsigma)'_{+,1}$ are defined as follows:

$$\boxed{\omega_\hbar \in \mathrm{Reg}_\hbar(\mathscr{T},\varsigma) \text{ iff } \forall f \in \mathscr{T}\ ,\ \lambda \mapsto \omega_\hbar(W_\hbar(\lambda f)) \text{ is a continuous map}}$$

- Let us define the *noncommutative Fourier transform* of a quantum state as

$$\hat{\omega}_\hbar(f) := \omega_\hbar(W_\hbar(f)) \ .$$

Theorem (Noncommutative Bochner [I.E. Segal 61])

The noncommutative Fourier transform is a bijection between $\mathrm{Reg}_\hbar(\mathscr{T},\varsigma)$ *and the functions* $\mathscr{G}_\hbar : \mathscr{T} \to \mathbb{C}$ *satisfying the following properties:*

- $\mathscr{G}_\hbar(0) = 1$;
- $\sum_{j,k} \bar{\alpha}_k \alpha_j \mathscr{G}_\hbar(f_j - f_k) e^{i\pi^2\hbar\varsigma(f_j,f_k)} \geq 0$;
- $\mathscr{G}_\hbar\big|_F$ *is continuous for all* F *finite dimensional subspace of* $\mathscr{T}$.

Wigner measures

- Let us denote by $\mathfrak{F}(\mathscr{T}, \varsigma)$ the set of all finite dimensional *symplectic* subspaces of $(\mathscr{T}, \varsigma)$, and define $\boxed{C_{0,\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})} \subset \mathbb{AP}(\mathscr{T})$ to be the subalgebra $\overline{\bigcup_{F\in\mathfrak{F}(\mathscr{T},\varsigma)} \mathscr{F}^{-1}L^1(F)}$, where the closure is intended in the uniform norm (the C*-norm of $\mathbb{AP}(\mathscr{T})$).

Theorem (Cylindrical Riesz-Markov)

Let $\mathscr{M}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ be the set of all positive cylindrical measures, and $\mathscr{M}^{\mathbb{C}}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ the set of all complex cylindrical measures. Then

$$\Big(C_{0,\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})\Big)' \cong \mathscr{M}^{\mathbb{C}}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T}) \;;$$

and

$$\Big(C_{0,\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})\Big)'_+ \cong \mathscr{M}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T}) \;.$$

Furthermore, if ω_0 is a positive continuous linear functional with $\|\omega_0\|_{\mathbb{AP}(\mathscr{T})'} = m$, then $\omega_0 \cong \mathfrak{m}$ with $\mathfrak{m}(\mathscr{T}^\star) = m$.

- Now, let us fix $F \in \mathfrak{F}(\mathscr{T}, \varsigma)$, and a symbol $a(T) = \int_F \hat{a}_F(k)\gamma_k(T)\mathrm{d}k$, $\hat{a}_F \in \mathscr{S}(F)$. Its quantization $\mathfrak{q}_\hbar^{\mathscr{T}}(a)$ reads

$$\mathfrak{q}_\hbar^{\mathscr{T}}(a) = \int_F \hat{a}_F(k) W_\hbar(k)\mathrm{d}k \ .$$

Theorem (Abstract sharp Gårding inequality)

Let $a : \mathscr{T}^\star \to \mathbb{R}_+$ *be a* positive *symbol such that there exists* $F \in \mathfrak{F}(\mathscr{T}, \varsigma)$ *and* $\hat{a}_F \in \mathscr{S}(F)$ *such that* $a(T) = \int_F \hat{a}_F(k)\gamma_k(T)\mathrm{d}k$*. Then there exists* $C_F > 0$

$$\mathfrak{q}_\hbar^{\mathscr{T}}(a) \geq -C_F \hbar \ .$$

Remark

- The stronger *Fefferman-Phong inequality*

$$\mathfrak{q}_\hbar^{\mathscr{T}}(a) \geq -C_F \hbar^2$$

actually holds, but the sharp Gårding is enough for our purposes.

Theorem (Existence of cylindrical Wigner measures [Ammari-Nier 08, M.F. 18])

Let $\omega_\hbar \in \mathrm{Reg}_\hbar(\mathscr{T}, \varsigma)$. *Then there exist*

- *a generalized sequence* $\hbar_\beta \to 0$,
- *a cylindrical measure* $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ *with* $0 \leq \mathfrak{m}(\mathscr{T}^\star) \leq 1$ *– called a Wigner measure for* $\omega_\hbar$ *–*

such that:

- *for all* a *such that there exists* $F \in \mathfrak{F}(\mathscr{T})$ *and* $\hat{a}_F \in \mathscr{S}(F)$ *such that* $a(T) = \int_F \hat{a}_F(k)\gamma_k(T)\mathrm{d}k$, *we have*

$$\lim_{\hbar_\beta \to 0} \omega_{\hbar_\beta}\big(\mathbb{q}^{\mathscr{T}}_{\hbar_\beta}(a)\big) = \int^{\bullet}_{\mathscr{T}^\star} a(T)\mathrm{d}\mathfrak{m}(T)\Big(= \int_F \hat{a}_F(k)\hat{\mathfrak{m}}(k)\mathrm{d}k\Big) .$$

Remark

- ⚠ The total mass of a Wigner measure $\mathfrak{m}$ may be < 1 (even zero!) ⚠
  This is a well-known fact also in finite dimensional standard semiclassical analysis (and it has physical implications).
- We would like to have no loss of mass (and sufficient conditions for it to be ensured), thus obtaining $\mathrm{Reg}_{\hbar}(\mathscr{T}, \varsigma) \ni \omega_{\hbar_\beta} \longrightarrow \omega_0 \in \mathrm{Reg}_0(\mathscr{T}, \varsigma)$.

Proof of the theorem

- Let us consider the functional $\mathfrak{O}_\hbar : a \mapsto \omega_\hbar(\mathfrak{q}_\hbar^{\mathscr{T}}(a))$, for all $a(T) = \int_F \hat{a}_F(k)\gamma_k(T)\mathrm{d}k$. $\mathfrak{O}_\hbar$ is linear and bounded (with respect to the uniform norm of a). Since $\mathscr{S}(F)$ is dense in $L^1(F)$, $\mathfrak{O}_\hbar$ is a densely defined bounded and linear functional on $C_{0,\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ and thus it extends uniquely to $\mathfrak{O}_\hbar \in \Big(C_{0,\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})\Big)'$.
- $\{\mathfrak{O}_\hbar\ ,\ \hbar \in (0,1)\}$ is relatively compact in the weak-* topology, so by Banach-Alaoglu there is a generalized sequence $\hbar_\beta \to 0$ and a complex cylindrical measure $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}^{\mathbb{C}}(\mathscr{T}^\star, \mathscr{T})$ such that $\mathfrak{O}_{\hbar_\beta} \to \mathfrak{m}$ in the weak-* topology.
- It remains to prove that $\mathfrak{m}$ is positive. This is an immediate consequence of the sharp Gårding inequality.

$\square$

The convergence of Fourier transforms

Lemma

Let $\omega_\hbar \in \mathrm{Reg}_\hbar(\mathscr{T}, \varsigma)$ *be such that there exists* $\hbar_\beta \to 0$ *and* $\omega_0 \in \mathrm{Reg}_0(\mathscr{T}, \varsigma)$ *such that, pointwise,* $\hat{\omega}_{\hbar_\beta} \underset{\hbar_\beta \to 0}{\longrightarrow} \hat{\omega}_0$. *Then* $\mathfrak{m}_{\omega_0} \in \mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ *is a Wigner measure of* $\omega_\hbar$ *with no loss of mass.*

Proof.

$$\lim_{\hbar_\beta \to 0} \omega_{\hbar_\beta}(\mathbb{q}^{\mathscr{T}}_{\hbar_\beta}(a)) = \lim_{\hbar_\beta \to 0} \int_F \hat{a}_F(k) \hat{\omega}_{\hbar_\beta}(k) \mathrm{d}k = \int_F \hat{a}_F(k) \hat{m}_{\omega_0}(k) \mathrm{d}k$$

by dominated convergence. □

Are cylindrical measures truly the classical states?

Can't we restrict to $\mathscr{P}\big(\mathscr{T}^{\star}, \Sigma(\mathscr{T}^{\star}, \mathscr{T})\big)$?

Are cylindrical measures truly the classical states?

Can't we restrict to $\mathscr{P}(\mathscr{T}^\star, \Sigma(\mathscr{T}^\star, \mathscr{T}))$? $\boxed{\textit{No!}}$

Are cylindrical measures truly the classical states?

Can't we restrict to $\mathscr{P}(\mathscr{T}^\star, \Sigma(\mathscr{T}^\star, \mathscr{T}))$? $\boxed{\textit{No!}}$

Lemma ([M.F. 18])

$\mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ (also $\mathscr{M}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})_{\leq 1}$) is included in the set of all possible Wigner measures: for any cylindrical probability $\mathfrak{m} \in \mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ there exists a regular state $\omega_{\hbar,\mathfrak{m}} \in \mathrm{Reg}_\hbar(\mathscr{T}, \varsigma)$ such that, pointwise, $\hat{\omega}_{\hbar,\mathfrak{m}} \underset{\hbar\to 0}{\longrightarrow} \hat{\mathfrak{m}}$.

Are cylindrical measures truly the classical states?

Can't we restrict to $\mathscr{P}(\mathscr{T}^\star, \Sigma(\mathscr{T}^\star, \mathscr{T}))$? *No!*

Lemma ([M.F. 18])

$\mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ (also $\mathscr{M}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})_{\leq 1}$) is included in the set of all possible Wigner measures: for any cylindrical probability $\mathfrak{m} \in \mathscr{P}_{\mathrm{cyl}}(\mathscr{T}^\star, \mathscr{T})$ there exists a regular state $\omega_{\hbar,\mathfrak{m}} \in \mathrm{Reg}_\hbar(\mathscr{T}, \varsigma)$ such that, pointwise, $\hat{\omega}_{\hbar,\mathfrak{m}} \underset{\hbar\to 0}{\longrightarrow} \hat{\mathfrak{m}}$.

Remark

- Interacting Gibbs measures for nonlinear PDEs (*e.g.* Hartree) are *cylindrical probabilities*, very useful to study rough solutions (see Bourgain, Tzvetkov, Oh, ...).

  Such measures can be obtained as *the Wigner measures of quantum Gibbs states* of (non-relativistic, interacting) QFTs [Fröhlich-Knowles-Schlein-Sohinger 17-22, Lewin-Nam-Rougerie 15-21, Sohinger-Rout 23].

Thank you for the attention (I)