INTERACTING BOSONS
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JOINT WORK with :

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The problem that I would like to consider is the following

$$\left\{\frac{1}{i}\partial_t + H_N\right\}\psi = 0 \quad ; \quad \psi(t=0) := \psi_0$$

where the Hamiltonian is

$$H_N := -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N}\sum_{j<k} v_N(|x_j - x_k|)$$

The potential v_N has the form

$$v_N(\cdot) := N^{3\beta} v(N^{\beta}\cdot) \quad , \quad 0 \le \beta \le 1$$

wecall $\beta = 1$ the critical case

Assumptions on interacting potential

• I will assume that $v \geq 0$ is radial smooth, bounded and decaying fast enough at spatial infinity so that $v \in L^1$. An extra assumption later will be $v' \leq 0$.

$$
\begin{array}{l}
\psi(t, x_1, \ldots x_N) \quad \text{wave} - \text{function} \\
\text{Bosons means} \\
\psi(t, x_{\sigma(1)}, \ldots x_{\sigma(N)}) = \psi(t, x_1 \ldots x_N) \\
\text{for all } \sigma : \{1, 2 \ldots N\} \mapsto \{1, 2 \ldots N\} \\
\text{permutations}
\end{array}
$$

Evolution preserves this property. σ commutes with H_N. Assumptions on v

$$
\begin{array}{l}
v \in L^\infty \cap L^1 \\
v(r) \geq 0 \quad , \quad v'(r) \leq 0
\end{array}
$$

also we assume v is smooth. L^1 and positivity are needed the rest are for convenience.

• Solution :

$$\psi(t) = e^{-itH_N}\psi_0$$

ψ_0 initial state

We would like to understand the evolution under special circumstances, meaning data of the form

$$\psi_0(x_1 \dots x_N) \approx \prod_{j=1}^{N} \phi_0(x_j)$$

Guess :

$$\psi(t, x_1 \dots x_N) \approx \prod_{j=1}^{N} \phi(t, x_j) \quad ?$$

$\phi(t,x)$ mean field (refer to as condensate)

Can this be true? Can I write an equation for $\phi(t,x)$ (refer to as mean field evolution).

BBGKY Hierarchy : Marginals

$$\gamma_{N,k}(t, x_1 \dots x_k; y_1 \dots y_k) =$$
$$\int dz_{k+1} \dots dz_N$$
$$\left\{\overline{\psi}(t, y_1 \dots y_k; z_{k+1}, \dots z_N)\psi(t, x_1 \dots x_k; z_{k+1} \dots z_N)\right\}$$

They satisfy the system of coupled equations

$$\frac{1}{i}\partial_t \gamma_{N,k} - \left[\Delta, \gamma_{N,k}\right] + \left[B_N, \gamma_{N,k+1}\right] = 0 \quad k = 1, 2 \dots$$

and letting $N \to \infty \quad \gamma_{N,k} \to \gamma_k$

$$\frac{1}{i}\partial_t \gamma_k - \left[\Delta, \gamma_k\right] + \left[B, \gamma_{N,k+1}\right] = 0 \quad k = 1, 2 \dots$$

$$\frac{1}{i}\partial_t \phi - \Delta\phi + g|\phi|^2\phi = 0 \quad \text{closure !}$$

REF: Spohn, (Golse, Bardos) and Elgart, Erdos, Schlein, Yau in a series of papers, Klainerman and Machedon (uniqueness of the hierarchy).

More recent work, T. Chen, N. Parvolvic, N. Tzirakis.

X. Chen, J. Holmer, their work inspired ours.

Cubic NLS, comments :

The mean field must satisfy the cubic NLS (also called Gross-Pitaevskii)

$$
\begin{aligned}
&\frac{1}{i}\partial_t\phi - \Delta\phi + g|\phi|^2\phi = 0\\
&g = \int v \quad \text{if } 0 < \beta < 1\\
&g = 8\pi a \quad \text{if } \beta = 1\\
&a = \text{scattering length}
\end{aligned}
$$

Scattering length a is

$$
\begin{aligned}
&\Big(-\Delta + \frac{1}{2}v\Big)f = 0\\
&f \sim 1 - \frac{a}{|x|} \quad \text{for large } |x|
\end{aligned}
$$

If $v > 0$ then $g > 0$ and the equation is called defocusing. If $g < 0$ the equation is called focusing. In our case we obtain the defocusing NLS.

Some more comments will help understand later developments... For N large but finite the mean field evolution is governed by

$$\frac{1}{i}\partial_t\phi_N - \Delta\phi_N \pm \left(v_N * |\phi_N|^2\right)\phi_N = 0$$
$$\text{formally } : \phi_N \to \phi \quad \text{where}$$
$$\frac{1}{i}\partial_t\phi - \Delta\phi \pm g|\phi|^2\phi = 0$$
$$+ \text{ defocusing} \quad , \quad - \text{ focusing}$$

The solution ϕ_N exists globally in time, but there is a dichotomy. The focusing NLS blows up while the defocusing exists globally in time. This means convergence $\phi_N \to \phi$ is not entirely obvious. Actually the focusing case raises some interesting questions.

• Cubic NLS is $H^{\frac{1}{2}}$ critical in $3-d$ meaning the initial value problem is well posed only if initial data are in H^s for $s \geq 1/2$, the borderline case is very challenging. See : Keel, Tao, Visan, Killip and B. Dodson, after work by the I-team.

Strichartz :

The main tool for studying NLS are Strichartz estimates. We will use them later so...consider the linear problem

$$\left\{\frac{1}{i}\partial_t - \Delta\right\}\phi = f$$
$$\phi(0) = \phi_0 \text{ data}$$

Then we have the following family of estimates,

$$\|\phi\|_{L^p(dt)L^q(dx)} \leq C\|f\|_{L^{p'}(dt)L^{q'}(dx)} + \|\phi_0\|_{L^2}$$
$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \quad , \quad \text{where } p \geq 2$$
$$p = \infty \ , \ q = 2 \text{ energy (easy)}$$
$$p = 2 \ , \ q = 6 \ \ \text{end} - \text{point Keel, Tao (challenging)}$$

Here p' and q' are the dual of p, q respectively.

• The mixed norms are computed as follows (for example)

$$\|\psi\|_{L^2(dt)L^6(dx)} := \\ = \left(\int dt \left(\int dx |\psi(t,x)|^6\right)^{1/3}\right)^{1/2}$$

• The end-point Strichartz estimate is a celebrated result by Keel and Tao.

How can we use Strichartz in order to study the existence of solutions of the cubic NLS? Here is a rough outline : First we take a half derivative, the equation looks like

$$\frac{1}{i}\partial_t D^{1/2}\phi - \Delta D^{1/2}\phi \pm |\phi|^2 D^{1/2}\phi = 0$$

$$D^{1/2}f = \mathcal{F}^{-1}|\xi|^{1/2}\mathcal{F}\Big(f\Big)$$

we write $D^{\alpha} = \mathcal{F}^{-1}|\xi|^{\alpha}\mathcal{F}\Big(f\Big)$

also $\Big\langle D^{\alpha}\Big\rangle = \mathcal{F}^{-1}(|\xi|^{\alpha}+1)\mathcal{F}\Big(f\Big)$

- Leibnitz for fractional derivatives:

$$D^{\alpha}(fg) \sim (D^{\alpha}f)g + f(D^{\alpha}g)$$

as far as L^p estimates are concerned

Coifman, Meyer, Kenig Ponce, Vega, Christ, Weinstein. Thus

$$D^{1/2}(|\phi|^2\phi) \sim 2|\phi|^2(D^{1/2}\phi) + (D^{1/2}\overline{\phi}))\phi^2$$

all terms are treated similarly.

Applying the end point Strichartz we have

$$\|D^{1/2}\phi\|_{L^2(dt)L^6(dx)} \leq_C \|\phi_0\|_{H^{1/2}(dx)}$$
$$+\|\phi^2(D^{1/2}\phi)\|_{L^2(dt)L^{6/5}(dx)}$$

and

$$\|\phi^2(D^{1/2}\phi)\|_{L^2(dt)L^{6/5}(dx)}$$
$$\leq_C \|\phi\|^2_{L^\infty(dt)L^3(dx)}\|D^{1/2}\phi\|_{L^2(dt)L^6(dx)}$$
$$\leq_C \|\phi\|^2_{L^\infty(dt)H^{1/2}(dx)}\|D^{1/2}\phi\|_{L^2(dt)L^6(dx)}$$

where we used Sobolev etc. The point here is that

$$\|\phi\|_{L^\infty(dt)H^{1/2}(dx)}$$

is the energy estimate in the class of Strichartz i.e. we can put on the right hand side of the inequality the quantity

$$\sup_{p,q}\|D^{1/2}\phi\|_{L^p(dt)L^q(dx)}$$

Once we realize that we can **balance** the inequalities i.e. the norms appearing on the right hand side also appear on the left we can prove **local existence** on some time interval using standard arguments. (Fixed point)

Global (in time) existence of solutions requires the knowledge of some apriori control of norms through (for example) energy type estimates. The energy for the cubic NLS is

$$\int_{\mathbb{R}^3} dx \left\{ |\nabla \phi|^2 \pm g|\phi|^4 \right\}$$

Notice that the defocusing case allows us the control the H^1 norm which however is of no direct use since we are considering $H^{1/2}$ data. This "explains" why proving global existence at low regularity is challenging. See B. Dodson... This appears as a Mathematical game but sometimes you are forced to play it.

Idea : Good estimates for the mean field PDE lead to good estimates for the many body problem.

Second Quantization, Fock space

Our original motivation came from paper by Rodnianski, Schlein. They used coherent states and Fock space formalism. The Fock space is a Hilbert type of space,

$$\mathbb{F} = \mathbb{F}_0 \oplus \mathbb{F}_1 \oplus \ldots \oplus \mathbb{F}_N \oplus \ldots$$
$$\mathbb{F}_n := L^2(\mathbb{R}^{3n}) \quad n-th \text{ sector}$$
$$\mathbb{F}_0 = \mathbb{C} \ \text{ complex constants}$$
$$\Big|\psi\Big\rangle = (\psi_0 \ldots \psi_n \ldots) \text{ vector in } \mathbb{F}$$
$$\Big|0\Big\rangle := (1, 0 \ldots) \quad \text{vacuum}$$

We have : Creation and annihilation operators defined with the property

$$\int a_x \overline{f}(x) : \mathbb{F}_{n+1} \mapsto \mathbb{F}_n$$
$$\int a_x^* f(x) : \mathbb{F}_{n-1} \mapsto \mathbb{F}_n$$
$$\text{for } \ f \in L^2(\mathbb{R}^3)$$

- Idea is that they create or destroy particles.

• For Bosons we symmetrize the functions i.e. we consider

$$L^2_{\mathsf{symm}}(\mathbb{R}^{3n})$$

which are functions invariant under coordinate permutations.

Defined as follows :

(1) $a_x(\psi_{n+1}) := \sqrt{n+1}\psi_{n+1}([x], x_1 \dots x_n)$

$[x]$ means frozen

(2) $a^*_x(\psi_{n-1})$

$$:= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j)\psi_{n-1}(x_1 \dots x_{j-1}, x_{j+1} \dots x_{n+1})$$

insert delta function at every "slot"

They satisfy the commutation relation (easily checked)

$$\left[a_x, a^*_y\right] = \delta(x, y)$$

Now we can write a Fock space Hamiltonian and consider the Fock space evolution. The evolution on the sector $\mathbb{F}_N$ is the "classical" original evolution as before.

• Fock space allows us to introduce (and use) Algebra techniques into the problem.

The evolution of the vector $\Big|\psi\Big\rangle$ is described by,

$$\left\{\frac{1}{i}\partial_t + \mathcal{H}_N\right\}\Big|\psi\Big\rangle = 0$$

where

$$\mathcal{H}_N :=$$
$$\int -a_x^*\Delta_x\delta(x,y)a_y + \frac{1}{N}a_x^*a_y^*v_N(x-y)a_xa_y$$

In the Fock space picture the number of particles is not fixed. A vector $\Big|\psi\Big\rangle$ carries arbitrarily large number of particles (in general) and the average number of particles is measured using the number operator

$$\mathcal{N} := \int a_x^* a_x$$

so that

$\Big\langle\psi\Big|\mathcal{N}\Big|\psi\Big\rangle = N$ number of particles

The Hamiltonian commutes with the number operator hence the average number of particles is conserved.

- We use **calligraphic letters** in order to denote Fock space operators.

Weil, Shale Segal, Bogoliubov

• Lie Algebra isomorphism. Consider symplectic matrices and the associated quadratic forms

$$K := \begin{pmatrix} -d^T(x,y) & \bar{l}(x,y) \\ k(x,y) & d(x,y) \end{pmatrix}$$

$$\mathcal{Q}(K) := \int dxdy \left\{ \begin{pmatrix} a_x, a_x^* \end{pmatrix} \begin{pmatrix} -d^T & \bar{l} \\ k & d \end{pmatrix} \begin{pmatrix} -a_y^* \\ a_y \end{pmatrix} \right\}$$

then :

$$\Big[\mathcal{Q}(K), \mathcal{Q}(L)\Big] = \mathcal{Q}\Big([K, L]\Big)$$

i.e. the commutation of quadratic forms is "equivalent" to the commutation of "matrices". The matrices here are 2×2 with entries kernels, hence infinite dimensional.

• Bogoliubov used it to "rotate" hence diagonalize quadratic Hamiltonians. (see Fetter) The idea was to compute the phonon spectrum of a superfluid. See also Lee, Huang and Yang.

Coherent states and pairs

Introduce the operators,

$$\mathcal{A}(\phi(t)) := \int_{\mathbb{R}^3} dx \left\{ \overline{\phi}(t,x) a_x - \phi(t,x) a_x^* \right\}$$

$$\mathcal{B}(k(t)) := \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 dx_2$$

$$\left\{ \overline{k}(t, x_1, x_2) a_{x_1} a_{x_2} - k(t, x_1, x_2) a_{x_1}^* a_{x_2}^* \right\}$$

We use them as follows

$$e^{-\sqrt{N}\mathcal{A}(\phi)} \Big| 0 \Big\rangle = \text{coherent state}$$

$$= \Big(\dots c_N \prod_{j=1}^{N} \phi(x_j) \dots \Big) \text{ at Nth entry}$$

and pair correction :

$$e^{-\mathcal{B}(k)} e^{-\sqrt{N}\mathcal{A}(\phi)} \Big| 0 \Big\rangle$$

i.e. we create states by applying certain type of operators to the vacuum. The elementary model is the Harmonic oscillator.

Our original motivation came from a paper by Rodnianski and Schlein where they used coherent states in order to estimate the rate of convergence of an exact solution which starts from a coherent state to the coherent state evolved via cubic NLS (Gross-Pitaevskii). The comparison was done in trace norms.

$$e^{\mathcal{A}(\phi(t))}\left(e^{it\mathcal{H}}e^{-\mathcal{A}(\phi(0))}\right)\Big|0\Big\rangle$$

Notice that we start with data $\phi(0)$ evolve through $e^{it\mathcal{H}}$ and "conjugate" via $e^{-\mathcal{A}(\phi(t))}$.

By adding the correction due to $e^{-\mathcal{B}(k(t,x,y))}$ we can obtain estimates in Fock space norm (L^2), however one must figure out the evolution of "kernel" $k(t, x_1, x_2)$. We can think of k as the kernel of an operator or simply as a function.

Some remarks are in order. The exponential of the matrix

$$K := \begin{pmatrix} 0 & \overline{k} \\ k & 0 \end{pmatrix}$$

can be computed,

$$\exp \begin{pmatrix} 0 & \overline{k} \\ k & 0 \end{pmatrix} = \begin{pmatrix} \cosh(k) & \overline{\sinh(k)} \\ \sinh(k) & \cosh^T(k) \end{pmatrix} := \begin{pmatrix} c & \overline{u} \\ u & c^T \end{pmatrix}$$

The hyperbolic functions appearing above are computed via composition products.

$$\sinh(k) = k + \frac{1}{3} k \circ \overline{k} \circ k + \ldots$$

where for example

$$(k \circ \overline{k})(x_1, x_2) := \int dy \left\{ k(x_1, y) \overline{k}(y, x_2) \right\}$$

For later use we adopt the notation,

$$c := \cosh(k) \quad , \quad u := \sinh(k)$$

$$\psi_p := \sinh(2k) = 2u \circ c$$

$$\gamma_p := \cosh(2k) - 1$$

Comparing the exact with the approximate dynamics

Now the idea is as follows :

$$\left|\psi_{\text{exact}}(t)\right\rangle = e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi(0)} e^{-\mathcal{B}(k(0)} \left|0\right\rangle$$

$$\left|\psi_{\text{approx}}(t)\right\rangle := e^{-\sqrt{N}\mathcal{A}(\phi(t)} e^{-\mathcal{B}(k(t)} \left|0\right\rangle$$

$$\left|\psi_{\text{red}}(t)\right\rangle$$

$$:= e^{\sqrt{N}\mathcal{A}(\phi(t)} e^{\mathcal{B}(k(t)} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi(0)} e^{-\mathcal{B}(k(0)} \left|0\right\rangle$$

Call

$\mathcal{M}(t) := e^{-\sqrt{N}\mathcal{A}(\phi(t)} e^{-\mathcal{B}(k(t)}$ Unitary operator

A computation leads to

$$\frac{1}{i}\partial_t \left|\psi_{\text{red}}\right\rangle = \mathcal{H}_{\text{red}} \left|\psi_{\text{red}}\right\rangle$$

where : $\mathcal{H}_{\text{red}} := \frac{1}{i}\left(\partial_t \mathcal{M}^*\right)\mathcal{M} + \mathcal{M}^*\mathcal{H}\mathcal{M}$

REMARK :

The operators a_x, a_x^* are unbounded (they are also distribution valued). When we consider integrals involving a_x, a_x^* and in particular exponentials we need to be careful to make sure that the operators are well defined. In our case one can prove that the operators are unitary. Notice that we made sure that $\mathcal{A}(\phi)$ and $\mathcal{B}(k)$ are skew adjoint. For example an operator of the form

$$\exp\left(\frac{1}{2}\int dxdy\left\{k(x,y)a_x^*a_y^*\right\}\right)$$

may not be well defined.

• $\mathcal{H}_{\text{red}}$ can be computed explicitly. The computation is tedious but straight forward. Our goal is to figure out the rule of evolution of ϕ, k. Start with the observation,

$$\mathcal{H}_{\text{red}} := \frac{1}{i}\Big(\partial_t \mathcal{M}^*\Big)\mathcal{M} + \mathcal{M}^*\mathcal{H}\mathcal{M}$$

rewrite it as :

$$-\frac{1}{i}\partial_t \mathcal{M}^* = \mathcal{M}^*\mathcal{H} - \mathcal{H}_{\text{red}}\mathcal{M}^*$$

and of course

$$\frac{1}{i}\partial_t \mathcal{M} = \mathcal{H}\mathcal{M} - \mathcal{M}\mathcal{H}_{\text{red}}$$

and consider the "matrices" defined below

$$\mathcal{L}_{m,n}(t, y_1 \dots y_m; x_1 \dots x_n)$$
$$:= \frac{1}{N^{(n+m)/2}} \Big\langle 0 \Big| \mathcal{M}^*(t) a^*_{y_1} \dots a^*_{y_m} a_{x_1} \dots a_{x_n} \mathcal{M}(t) \Big| 0 \Big\rangle$$
where $\mathcal{P} := a^*_{y_1} \dots a^*_{y_m} a_{x_1} \dots a_{x_n}$ monomial

We know the rule of evolution of $\mathcal{M}(t)$ so differentiating etc we find,

$$\frac{1}{i}\partial_t \mathcal{L} = \frac{1}{N^{(m+n)/2}} \times$$
$$\Big\{ \Big\langle 0 \Big| \Big[\mathcal{H}_{\mathrm{red}}, \mathcal{M}^* \mathcal{P} \mathcal{M} \Big] \Big| 0 \Big\rangle + \Big\langle 0 \Big| \mathcal{M}^* \Big[\mathcal{H}, \mathcal{P} \Big] \mathcal{M} \Big| 0 \Big\rangle \Big\}$$

Recall that we know (can compute)

$$\mathcal{H}_{\mathrm{red}} \Big| 0 \Big\rangle = \Big(X_0, X_1, X_2, X_3, X_4, 0 \dots \Big)$$
choose : $X_1 = X_2 = 0$

i.e. our rule that defines the evolution is the elimination of the first and second entries in $\mathcal{H}_{\mathrm{red}} \Big| 0 \Big\rangle$. This is natural. Since we have two functions at our disposal we should try to eliminate two terms.

In order to motivate our criterion let us consider the reduced evolution

$$\frac{1}{i}\partial_t\left|\psi_{\mathrm{red}}\right\rangle = \mathcal{H}_{\mathrm{red}}\left|\psi\right\rangle \quad , \quad \left|\psi_{\mathrm{red}}(0)\right\rangle = \left|0\right\rangle$$

Write

$$\left|\psi_{\mathrm{red}}\right\rangle = \left|\widetilde{\psi}\right\rangle + \left|0\right\rangle$$

and

$$\frac{1}{i}\partial_t\left|\widetilde{\psi}\right\rangle = \mathcal{H}_{\mathrm{red}}\left|\widetilde{\psi}\right\rangle + \mathcal{H}_{\mathrm{red}}\left|0\right\rangle$$

Therefore

$$\begin{aligned}\left|\widetilde{\psi}(t)\right\rangle &= \int_0^t e^{i(t-s)\mathcal{H}_{\mathrm{red}}}\mathcal{H}_{\mathrm{red}}\left|0\right\rangle ds \\ &= \int_0^t e^{i(t-s)\mathcal{H}_{\mathrm{red}}}\Big(X_0, X_1, X_2, X_3, X_4 \ldots\Big)\end{aligned}$$

The constant term X_0 can be absorbed as a phase.

Remark : $\mathcal{H}_{\mathrm{red}}$ does not preserve the number of particles. It contains terms (for example)

$$\int f_3 a^* a^* a^* + \text{c.c.}$$

The entries X_1, X_2, X_3, X_4 look like

$$
\begin{aligned}
X_1 &\sim O(\sqrt{N}) \\
X_2 &\sim O(1) \\
X_3 &\sim O\left(\frac{1}{\sqrt{N}}\right) \\
X_3 &\sim O\left(\frac{1}{N}\right)
\end{aligned}
$$

• assume that $X_1 = X_2 = 0$ then,

$$\text{if} \quad \mathcal{P} = \text{linear plus quadratic in } a, a^*$$
$$\text{then} \;:\; \left\langle 0 \middle| \left[\mathcal{H}_{\text{red}}, \mathcal{M}^* \mathcal{P} \mathcal{M} \right] \middle| 0 \right\rangle = 0$$

Hence the evolution simplifies to

$$\frac{1}{i} \partial_t \mathcal{L} = \frac{1}{N^\alpha} \left\langle 0 \middle| \mathcal{M}^* \left[\mathcal{P}, \mathcal{H} \right] \mathcal{M} \middle| 0 \right\rangle$$

Proof : The basic calculation is

$$\mathcal{M}^* a_x \mathcal{M} = \int dy \left\{ a_y c(y,x) + a_y^* u(y,x) \right\} + \sqrt{N} \overline{\phi}(x)$$
$$\mathcal{M}^* a_x^* \mathcal{M} = \int dy \left\{ a_y \overline{u}(y,x) + a_y^* \overline{c}(y,x) \right\} + \sqrt{N} \phi(x)$$

or

$$\mathcal{M}^* a_x \mathcal{M} := b_x + \sqrt{N} \overline{\phi}(x)$$
$$\mathcal{M}^* a_x^* \mathcal{M} := b_x^* + \sqrt{N} \phi(x)$$

This means that $\mathcal{M}$ transforms a monomial as follows,

$$\mathcal{M}^*\mathcal{P}(a^*, a)\mathcal{M} = \mathcal{P}\Big(b^* + \sqrt{N}\overline{\phi}, b + \sqrt{N}\phi\Big)$$
$$\mathcal{M}^*\mathcal{P}(a^*, a)\mathcal{M}\Big|0\Big\rangle = (f_0, f_1, f_2, 0 \ldots)$$

Recal that we also choose,

$$\mathcal{H}_{\text{red}}\Big|0\Big\rangle = \Big(X_0, 0, 0, X_3, X_4, 0 \ldots\Big)$$

Now it is easy to see why

$$\Big\langle 0\Big|\Big[\mathcal{H}_{\text{red}}, \mathcal{M}^*\mathcal{P}\mathcal{M}\Big]\Big|0\Big\rangle = 0$$

Since we have

$$\frac{1}{i}\partial_t \mathcal{L} = \frac{1}{N^a}\Big\langle 0 \Big| \mathcal{M}^* \Big[\mathcal{H}, \mathcal{P}\Big] \mathcal{M} \Big| 0 \Big\rangle$$

we need to compute the commutations $[\mathcal{H}, \mathcal{P}]$ for $\mathcal{P}$ linear and quadratic. Straight forward,

$$\Big[a_{x_1}, \mathcal{H}\Big] = \left(\Delta_{x_1} - \frac{1}{N}\int dy \left\{v_N(x_1 - y)a_y^* a_y\right\}\right) a_{x_1}$$

$$\Big[a_{x_1}a_{x_2}, \mathcal{H}\Big] = \left(\Delta_{x_1} - \frac{1}{2N}v_N(x_1 - x_2)\right) a_{x_1}a_{x_2}$$

$$+ a_{x_1}a_{x_2}\left(\Delta_{x_2} - \frac{1}{2N}v_N(x_1 - x_2)\right)$$

$$- \frac{1}{N}\int dy \left\{\Big(v_N(x_1 - y) + v_N(y - x_2)\Big)a_y^* a_y\right\} a_{x_1}a_{x_2}$$

$$\Big[a_{x_1}^* a_{x_2}, \mathcal{H}\Big] = a_{x_1}^* a_{x_2}\Delta_{x_2} - \Delta_{x_1}a_{x_1}^* a_{x_2}$$

$$+ \frac{1}{N}\int dy \left\{\Big(v_N(x_1 - y) - v_N(y - x_2)a_{x_1}^* a_y^* a_y a_{x_2}\right\}$$

Notice the presence of cubic and quartic monomials.

We obtain the "hierarchy"

$$
\begin{aligned}
&\left\{\frac{1}{i}\partial_t - \Delta_{x_1}\right\}\mathcal{L}_{0,1} \\
&+\frac{1}{N^{3/2}}\int dy \left\{v_N(x_1-y)\mathcal{L}_{1,2}(y;y,x_1)\right\} = 0 \\
&\left\{\frac{1}{i}\partial_t - \Delta_{x_1} - \Delta_{x_2}\right\}\mathcal{L}_{0,2} \\
&+\frac{1}{N^2}\int dy \left\{\Big(v_N(x_1-y) + v_N(y-x_2)\Big)\mathcal{L}_{1,3}\right\} = 0 \\
&\left\{\frac{1}{i}\partial_t + \Delta_{x_1} - \Delta_{x_2}\right\}\mathcal{L}_{1,1} \\
&+\frac{1}{N^2}\int dy \left\{\Big(v_N(x_1-y) - v_N(y-x_2)\Big)\mathcal{L}_{2,2} = 0\right\}
\end{aligned}
$$

The point here is that the system is closed. Our ansätz allows us to compute higher order matrices in terms of $\mathcal{L}_{0,1}$, $\mathcal{L}_{0,2}$, $\mathcal{L}_{1,1}$ and their conjugates.

We have to compute the $\mathcal{L}$ matrices...

$$\mathcal{L}_{0,1} = \Big\langle 0 \Big| b_{x_1} + \sqrt{N}\phi(x_1) \Big| 0 \Big\rangle = \sqrt{N}\phi(x_1)$$
$$\mathcal{L}_{0,2} =$$
$$= \Big\langle 0 \Big| \Big(b_{x_1} + \sqrt{N}\phi(x_1)\Big)\Big((b_2 + \sqrt{N}\phi(x_2)\Big)\Big| 0 \Big\rangle$$
$$= N\phi(x_1)\phi(x_2) + (u \circ c)(x_1, x_2)$$
$$:= N\Big(\psi_c + \frac{1}{2N}\psi_p\Big) := N\psi$$
$$\mathcal{L}_{1,1} = N\overline{\phi}(x_1)\phi(x_2) + (\overline{u} \circ u)(x_1, x_2)$$
$$= N\Big(\gamma_c + \frac{1}{2N}\gamma_p\Big) := N\gamma$$
$$c := \text{condensate} \quad , \quad p := \text{pairs}$$

as a matter of fact,

$$\gamma_p = \cosh(2k) - 1 \quad , \quad \psi_p = \sinh(2k)$$

The rest of the matrices $\mathcal{L}_{1,3}$ etc can be computed in a similar (tedious) manner. The ansätz that we adopted "closes" the hierarchy so we only have (essentially) two equations.

Matrices again..this is inspired by the Lie Algebra isomorphism (see earlier) Recall

$$J \begin{pmatrix} a \\ a^* \end{pmatrix} = \begin{pmatrix} -a^* \\ a \end{pmatrix} \quad , \quad J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and form the tensor product,

$$\begin{pmatrix} -a_x^* \\ a_x \end{pmatrix} (a_y a_y^*) = \begin{pmatrix} -a_x^* a_y & -a_x^* a_y^* \\ a_x a_y & a_x a_y^* \end{pmatrix}$$

Therefore we construct the 2×2 matrix which we call Φ,

$$\Phi := \begin{pmatrix} -\gamma & -\overline{\psi} \\ \psi & \gamma^T \end{pmatrix} = \Phi_c + \Phi_p$$

$$= \begin{pmatrix} -\overline{\phi} \otimes \phi & -\overline{\phi} \otimes \overline{\phi} \\ \phi \otimes \phi & \phi \otimes \overline{\phi} \end{pmatrix} + \frac{1}{2N} \begin{pmatrix} -\gamma_p & -\overline{\psi}_p \\ \psi_p & \gamma_p^T \end{pmatrix}$$

i.e. we can separate the matrix into condensate plus pairs. The idea is that (for example) $\psi(t, x_1, x_2)$ should not be (in general) a tensor product thus describing pair interactions.

Let us introduce the matrix

$$S_3 := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

and the operator,

$$H_\rho = -\Delta_x + (v_N * \rho)(x)$$

where ρ is the overall **one particle density**,

$$\rho(t,x) := \gamma(t,x,x) = \rho_c + \rho_p$$
$$\rho_c(t,x) = |\phi(t,x)|^2$$
$$\rho_p(t,x) = \frac{1}{2N} \int_{\mathbb{R}^3} dy \left\{ |u(t,x,y)|^2 \right\}$$

As we will see the average of the density is a conserved quantity.

The adjoint of the matrix Φ is

$$\Phi^* = S_3 \Phi S_3 = \begin{pmatrix} -\gamma & \overline{\psi} \\ -\psi & \gamma^T \end{pmatrix} \quad \gamma^T = \overline{\gamma}$$

With these we have the tools to write down the evolution equations for Φ and Φ_c. The equations for the matrices Φ, Φ_c are,

$$\frac{1}{i}\partial_t \Phi + \Big[H_\rho S_3, \Phi\Big] + \frac{1}{2N}\Big[S_{3,} v_N \Phi\Big] + \Big[v_N \Phi^*, \Phi\Big]$$
$$= \Big[v_N \Phi_c^*, \Phi_c\Big]$$
$$\frac{1}{i}\partial_t \Phi_c + \Big[H_\rho S_3, \Phi_c\Big] + \Big[v_N \Phi^*, \Phi_c\Big]$$
$$= \Big[v_N \Phi_c^*, \Phi_c\Big]$$

From the above we can see that if $\Phi_p = 0$ then the second equation becomes

$$\frac{1}{i}\partial_t \Phi_c + \Big[H_{\rho_c} S_3, \Phi_c\Big] = 0$$

which is the cubic NLS (Gross Pitaevskii).

A few comments about our notation, $v_N\Phi$ means point wise multiplication i.e.

$$v_N\Phi := \begin{pmatrix} -v_N\gamma & -v_N\overline{\psi} \\ v_N\psi & v_N\gamma^T \end{pmatrix}$$

Commutations are computed as composition products, thus

$$\left[v_N\Phi^*, \Phi\right] = (v_N\Phi^*) \circ \Phi - \Phi \circ (v_N\Phi^*)$$
$$= \int dy \,\{v_N(x_1 - y)\Phi^*(x_1, y)\Phi(y, x_2)\}$$
$$- \int dy \,\{\Phi(x_1, y)v_N(y - x_2)\Phi^*(y, x_2)\}$$

It is a good idea to separate Φ into symmetric and anti-symmetric parts i.e. we define

$$\Gamma := \frac{1}{2}\left(\Phi + \Phi^*\right) = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma^T \end{pmatrix}$$
$$\Lambda := \frac{1}{2}\left(\Phi - \Phi^*\right) = \begin{pmatrix} 0 & -\overline{\psi} \\ \psi & 0 \end{pmatrix}$$

Note that $\gamma^T = \overline{\gamma}$ while ψ is symmetric i.e. $\psi^T = \psi$. Now the evolution equations can be written as follows

$$\frac{1}{i}\partial_t \Gamma + \left[H_\rho S_3, \Gamma\right] + \left[v_N \Gamma, \Gamma\right] - \left[v_N \Lambda, \Lambda\right] = F(\phi)$$
$$\frac{1}{i}\partial_t \Lambda + \left[H_\rho S_3, \Lambda\right] + \frac{1}{N} v_N \Lambda + \left[v_N \Gamma, \Lambda\right] - \left[v_N \Lambda, \Gamma\right]$$
$$= G(\phi)$$

where $F(\phi)$ and $G(\phi)$ are quartic expressions of the condensate ϕ. Notice the presence of the term of

$$\frac{1}{N} v_N \Lambda$$

in the Λ equation.

This system was also derived by V. Back, S. Breteaux, T. Chen, J. Fröhlich, I.M. Sigal in a slightly different context. See also Nam and Napiorkowski.

We had (with M. Machedon) an earlier derivation of the coupled system which used a different idea. Recall the computation

$$\mathcal{H}_{\text{red}}\Big|0\Big\rangle = (X_0, X_1, X_2, X_3, X_4, 0\ldots)$$

The first term (constant) X_0 integrated over time serves as Lagrangian whose variation produces the evolution equations. Let us call

$$L(\phi, u) := -\int dt\{X_0\}$$

where ϕ and $u = \sinh(k)$ are the dynamic variables.

We can compute (tedious but straight forward). With the convention $\phi(t, x_1) = \phi_1$, $u(t, x_1, x_2) = u_{1,2}$ etc we have

$$
\begin{aligned}
&L =: -\int dt\{X_0\} \\
&= N \int dt dx_1 \left\{ \mathrm{Im}(\overline{\phi}_1 \partial_t \phi_1) + |\nabla \phi_1|^2 \right\} \\
&+ \int dt dx_1 dx_2 \left\{ \mathrm{Im}(\overline{u}_{1,2} \partial_t u_{1,2}) + |\nabla u_{1,2}|^2 \right\} \\
&+ \frac{N}{2} \int dt dx_1 dx_2 \left\{ v_{N,1-2} \left| \phi_1 \phi_2 + \frac{1}{2N} \psi_{p,1,2} \right|^2 \right\} \\
&+ \frac{1}{2} \int dt dx_1 dx_2 dx_3 \left\{ v_{N,1-2} \Big| \phi_1 u_{2,3} + \phi_2 u_{1,3} \Big|^2 \right\} \\
&+ \frac{1}{2N} \times \\
&\int dt dx_1 dx_2 \left\{ v_{N,1-2} |(u \circ \overline{u})_{1,2}|^2 + (u \circ \overline{u})_{1,1} (\overline{u} \circ u)_{2,2} \right.
\end{aligned}
$$

The variation

$$
\frac{\delta L}{\delta \overline{\phi}} = 0 \quad , \quad \frac{\delta L}{\delta \overline{u}} = 0
$$

implies $X_1 = X_2 = 0$

The resulting system can be rewritten in the form that we gave earlier.

The importance of the Lagrangian formulation lies in the fact that one can derive conservation laws (Nöther's theorem). Time translation invariance leads to energy conservation. Gauge invariance leads to mass (or number) conservation. Momenta are not conserved but structure equations can be derived using Lie derivatives of appropriate vector fields.

• The "naive" method is to multiply the equation with an appropriate derivative and integrate by parts. This, of course, works but we do not know why. An even more direct method is to differentiate in time some (appropriate) integral.

The energy is given by the integral, (again we denote $u_{1,2} := u(t, x_1, x_2)$ etc)

$$
\begin{aligned}
NE &:= N \int dx_1 \Big\{ |\nabla \phi_1|^2 \Big\} + \int dx_1 dx_2 \Big\{ |\nabla u_{1,2}|^2 \Big\} \\
&+ \frac{N}{2} \int dx_1 dx_2 \left\{ v_{N,1-2} \left| \phi_1 \phi_2 + \frac{1}{2N} \psi_{p,1,2} \right|^2 \right\} \\
&+ \frac{1}{2} dx_1 dx_2 dx_3 \left\{ v_{N,1-2} |\phi_1 u_{2,3} + \phi_2 u_{1,3}|^2 \right\} \\
&+ \frac{1}{2N} \int dx_1 dx_2 \left\{ v_{N,1-2} \Big(|(u \circ \overline{u})_{1,2}|^2 + \rho_{p,1} \rho_{p,2} \Big) \right\}
\end{aligned}
$$

where ρ_p is the pair density density

$$\rho_{p,1} := \gamma_p(t, x_1, x_1)$$

The integral of the overall density $\rho(t, x, x) := \gamma(t, x, x)$ is conserved.

$$M := \int dx \{\rho\} = \text{constant}$$

Recall that $\rho = \rho_c + \rho_p$ with $\rho_c = |\phi|^2$ and $\rho_p = (1/2N)\gamma_p(t, x, x)$.

• Interaction Morawetz estimate (Lin and Srauss) (original idea from I-team Colliander, Keel, Staffilani, Takaoka, Tao) here due to a very nice argument by Jacky Chong. The estimate is

$$\|\rho^2\|_{L^1(dtdx)} \leq_C \sqrt{EM}$$

The point of this and the other energy, mass estimates is that the hold globally in time. It is the extra ingredient that goes into Strichartz estimates.

General idea is to consider :

$p_j :=$ momenta , $\rho :=$ density , $e :=$ energy

which satisfy a set of structure equations

$$\begin{aligned}
&\partial_t \rho + \nabla_j p^j = 0 \\
&\partial_t p_j + \nabla_k \left\{ \sigma^k{}_j - \delta^k{}_j \lambda \right\} + l_j = 0 \\
&\partial_t e + \nabla_j q^j = 0 \\
&\lambda = \text{Lagrangian density}
\end{aligned}$$

and for appropriate vector field X^j the tensor product contracted with X^j

$$\partial_t \left((\rho \otimes p_j) X^j \right) = \ldots$$

The end result is to check that the signs of various quantities are desirable. For the defocusing NLS the signs cooperate.

EXISTENCE

We need a general method to establish existence of our system. The challenge is to pick the correct norms so that we can balance the left with the right hand side. See our earlier computation with the cubic NLS.

Motivated by the cubic NLS we use Strichartz with half derivatives on each variable

$$D_1^{1/2}\Phi(t,x_1,x_2)D_2^{1/2}$$

For $\beta < 1$ we have some room to consider higher derivatives,

$$D_1^{\alpha}\Phi(t,x_1,x_2)D_2^{\alpha} \quad , \qquad \text{where } \alpha = \frac{1+\epsilon}{2}$$

We will use mixed Strichartz norms, motivated by work of Xuwen Chen and Justin Holmer.

Remark :

Let me explain the reason for doing what will follow. First, for N fixed it is relatively easy to prove existence of solutions with apriori bounds that contain N. However we want to establish existence using apriori bounds that are independent of N for two reasons. One is that we would like to consider the limit equations (as $N \to \infty$. The other is that if we find bounds independent of N this in turn implies better bounds for the Fock space evolution of the general form

$$\left\| \left| \psi_{\text{approx}} \right\rangle - \left| \psi_{\text{exact}} \right\rangle \right\|_{\mathbb{F}} \leq \frac{C(t)}{N^{\alpha}}$$

where $C(t)$ grows like a polynomial in time. See E. Kuz, J. Chong and Z. Zhao as well (recent) J. Chong, M. Machedon, Z. Zhao.

Mixed Strichartz

The inspiration for this was the work of X. Chen and J. Holmer. Consider the linear equation,

$$\left\{\frac{1}{i}\partial_t - \Delta_1 - \Delta_2\right\}\psi = F \quad , \quad \psi(0) = \psi_0$$
$$x_j \in \mathbb{R}^3 \quad \text{for} \quad j = 1, 2$$

Then the following estimates hold

$$\|\psi\|_{L^p(dt)L^q(dx_1)L^2(dx_2)} \leq_C \|F\|_{L^{p'}(dt)L^{q'}(dx_1)L^2(dx_2)} + \|\psi_0\|_{L^2}$$

where

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \quad p \geq 2$$

Notice that in space coordinates we use mixed norms $L^q(dx_1)L^2(dx_2)$ and we attain the range of exponents as in the three dimensional case. Also the end-point estimate holds true. Another important observation is the any norm on the right hand side controls any other on the left.

Simple rotations show that (for the same range of exponents we have (reversing the roles of x_1 and x_2,

$$\|\psi\|_{L^p(dt)L^q(dx_2)L^2(dx_1)} \leq_C \|F\|_{L^{p'}(dt)L^{q'}(dx_2)L^2(dx_1)} + \|\psi_0\|_{L^2}$$

We also have the inequalities

$$\|\psi\|_{L^p(dt)L^q(dx_{1+2})L^2(dx_{1-2})} \leq_C \|F\|_{L^{p'}(dt)L^{q'}(dx_{1+2})L^2(dx_{1-2})} + \|\psi_0\|_{L^2}$$

where the rotated coordinates are,

$$x_{1+2} := \frac{1}{\sqrt{2}}(x_1 + x_2)$$
$$x_{1-2} := \frac{1}{\sqrt{2}}(x_1 - x_2)$$

The reason that we want to consider mixed spaces and rotated coordinates in is because (recall the equation for Λ)

$$\frac{1}{i}\partial_t\Lambda + \left[H_\rho S_3, \Lambda\right] + \frac{1}{N}v_N\Lambda + \left[v_N\Gamma, \Lambda\right] - \left[v_N\Lambda, \Gamma\right]$$
$$= G(\phi)$$

and observe

$$\frac{1}{N}v_N(x_{1-2})\Lambda(t, x_1, x_2)$$
$$= \frac{1}{N}v_N(x_{1-2})\Lambda(t, x_{1+2}, x_{1-2})$$

It is advantageous to estimate in rotated coordinates. On the other hand

$$\left[v_N\Gamma, \Lambda\right] = \int dy \left\{v_N(x_1 - y)\Gamma(x_1, y)\Lambda(y, x_2)\right\}$$
$$- \int dy \left\{\Lambda(x_1, y)v_N(y - x_2)\Gamma(y, x_2)\right\}$$
$$\to \quad \text{as } N \to \infty$$
$$\Big(\rho(x_1)S_3\Big)\Lambda(x_1, x_2) - \Lambda(x_1, x_2)\Big(\rho(x_2)S_3\Big)$$

and the rotated coordinates do not offer an advantage.

Here we have that ρ is the density,

$$\rho(t,x) = \gamma(t,x,x)$$

In a similar spirit,

$$\left[v_N \Lambda, \Gamma\right] \to \quad \text{as } N \to \infty$$
$$\Lambda(t,x_1,x_1)\Gamma(t,x_1,x_1) - \Gamma(t,x_1,x_2)\Lambda(t,x_2,x_2)$$

This indicates the need to estimate $\Lambda(t,x,x)$ or ψ since

$$\Lambda = \begin{pmatrix} 0 & -\overline{\psi} \\ \psi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\overline{\phi \otimes \phi} \\ \phi \otimes \phi & 0 \end{pmatrix} + \frac{1}{2N} \begin{pmatrix} 0 & -\overline{\psi}_p \\ \psi_p & 0 \end{pmatrix}$$

i.e. we collapse the coordinates along the diagonal. These are called "collapsing" estimates and the norm is

$$\| \cdot \|_{L^\infty(dx_{1-2})L^2(dtdx_{1+2})}$$

• Finally we would like to imitate the strategy for the cubic NLS where we took half a derivative. Now we need to estimate :

$$\|D_1^{1/2} \Phi D_2^{1/2}\|_{\mathcal{S}}$$
$\mathcal{S} =$ the family of Strichartz norms

Let us check if the method works by looking at some "judiciously" chosen terms and see if we can balance them. In what follows I will pretend that I took $N \to \infty$ in order to simplify the computations. Recall the equations,

$$\frac{1}{i}\partial_t \Gamma + \Big[H_\rho S_3, \Gamma\Big] + \Big[v_N \Gamma, \Gamma\Big] - \Big[v_N \Lambda, \Lambda\Big] = F(\phi)$$

$$\frac{1}{i}\partial_t \Lambda + \Big[H_\rho S_3, \Lambda\Big] + \frac{1}{N} v_N \Lambda + \Big[v_N \Gamma, \Lambda\Big] - \Big[v_N \Lambda, \Gamma\Big]$$
$$= G(\phi)$$

Differentiating the second equation we obtain something like (Fractional Leibnitz)

$$\frac{1}{i}\partial_t D_1^{1/2} \Lambda D_2^{1/2} + \Big[-\Delta S_3, \Lambda\Big] =$$
$$-\frac{1}{N}(D v_N(x_{1-2}))\Lambda(t, x_1, x_2)$$
$$-\rho(x_1) D_1^{1/2} \Lambda D_2^{1/2}$$
$$+(D_1^{1/2} \Lambda(t, x_1, x_1))(\Gamma(t, x_1, x_2) D_2^{1/2})$$
$$+\dots$$

I picked three terms that explain what comes next. First look at the second term in dual end-point Strichartz,

$$\left\| \rho(t, x_1) D_1^{1/2} \Lambda D_2^{1/2} \right\|_{L^2(dt) L^{6/5}(dx_1) L^2(dx_2)}$$
$$\leq \|\rho\|^{\theta}_{L^\infty(dt) L^1(dx)} \|\rho\|^{1-\theta}_{L^2(dtdx)} \times$$
$$\|D_1^{1/2} \Lambda D_2^{1/2}\|_{L^2(dt) L^6(dx_1) L^2(dx_2)}$$

So this is O.K. because of the "mass" conservation and the collapsing estimate. The exponents θ and $1-\theta$ can be computed but are irrelevant.

Now let us look at the third term,

$$\left\|D_1^{1/2}\Lambda(x_1,x_1)(\Gamma D_2^{1/2})\right\|_{L^2(dt)L^{6/5}(dx_1)L^2(dx_2)}$$
$$\le \|D_1^{1/2}\Lambda(t,x_1,x_1)\|_{L^2(dtdx_1)}\times$$
$$\|\Gamma D_2^{1/2}\|_{L^\infty(dt)L^3(dx_1)L^2(dx_2)}$$
$$\le \|D_1^{1/2}\Lambda(t,x_1,x_1)\|_{L^2(dtdx_1)}\times$$
$$\|\Gamma D_2^{1/2}\|_{L^\infty(dt)L^2(dx_2)L^3(dx_1)}$$
$$\le \|D_1^{1/2}\Lambda(t,x_1,x_1)\|_{L^2(dtdx_1)}\times$$
$$\|D_1^{1/2}\Gamma D_2^{1/2}\|_{L^\infty(dt)L^2(dx_1dx_2)}$$

where we used Sobolev and the last term is the energy estimate in the Strichartz family. The above calculation tells us that we need to estimate

$$\left\|D_{1+2}^{1/2}\Lambda(t,x_{1+2},x_{1-2})\right\|_{L^\infty(dx_{1-2})L^2(dtdx_{1+2})}$$

These are called (linear) collapsing estimates. The reason for the name is because we collapse the coordinates i.e. we take $x_1=x_2$.

Now let us look at the first term. First notice the

$$\frac{1}{N}(Dv_N) = \frac{1}{N}N^{4\beta}Dv(N^\beta x)$$

If $\beta = 1$ then this looks like

$$N^3 w(Nx)$$

which for all practical purposes is a delta function. This means that we can only estimate it in L^1. Let us put a delta function and write

$$\delta(x_{1-2})\Lambda(t, x_{1+2}, x_{1-2})$$

Unfortunately this does not belong to any dual Strichartz spaces so we need a new idea.

What saves the day is the use of time derivatives. Let us look at the Fourier transform of the term,

$$1(\xi_{1-2})\widehat{\Lambda}(\tau, \xi_{1+2})$$

Obviously the integrability in ξ_{1-2} is a problem. Now

$$\begin{aligned}
&\text{write } F = \delta(x_{1-2})\Lambda(t, x_{1+2}, x_{1-2}) \quad \text{and} \\
&D_1^{1/2}\Lambda D_2^{1/2} = \int_0^t e^{i(t-s)(\Delta_1+\Delta_2)} F(s,\cdot)ds \\
&= \int_0^t \frac{i}{\Delta_1+\Delta_2}\partial_s\left\{e^{i(t-s)(\Delta_1+\Delta_2)}\right\}F(s,\cdot)ds \\
&= \int_0^t e^{i(t-s)(\Delta_1+\Delta_2)}\frac{-i\partial_s F(s,\cdot)}{\Delta_{1+2}+\Delta_{1-2}}ds + \text{b.t.}
\end{aligned}$$

In Fourier space we now have

$$\frac{\partial_t\widehat{\Lambda}(t, \xi_{1+2}, \xi_{1+2})}{|\xi_{1+2}|^2 + |\xi_{1-2}|^2}$$

thus we gain integrability in ξ_{1-2}.

Actually we do not need a full time derivative, it is enough to consider only a quarter time derivative and estimate,

$$\left\| |\partial_t|^{1/4} \Lambda(t, x_{1+2}, x_{1-2}) \right\|_{L^\infty(dx_{1-2}) L^2(dt dx_{1+2})}$$

This leads us to a new ingredient namely quarter **time derivative** combined with collapse.

Collapsing estimates

These type of estimates appeared in the work of Klainerman and Machedon (the time derivative is a new ingredient). Consider the linear evolution,

$$\left\{\frac{1}{i}\partial_t - \Delta_1 - \Delta_2\right\}\Lambda = F \quad , \quad \Lambda(0) = \Lambda_0$$

Then we have

$$\begin{aligned}
&\|(|\partial_t|^{1/4} + D_{1+2}^{1/2})\Lambda\|_{L^\infty(dx_{1-2})L^2(dtdx_{1+2})} \\
&\leq_C \|D_1^{1/2}\Lambda_0 D_2^{1/2}\|_{L^2} \\
&+\|D_1^{1+/2} F D_2^{1+/2}\|_{X^{-1/2+\delta}} \\
&+\|D_1^{1+/2} F\|_{X^{-1/4+\delta}}
\end{aligned}$$

where X^b are Bourgain spaces.

Bourgain spaces

In the context of Schrödinger they were introduced and used by Bourgain. For the wave equation they were introduced (earlier) by M. Beals but their first substantial use was by Klainerman and Machedon.

The symbol of the Schrödinger operator is (Fourier transform in space-time)

$$\sigma := \tau + |\xi_1|^2 + |\xi_2|^2$$
$$\left\langle \sigma \right\rangle := |\sigma| + 1$$
$$\|F\|_{X^b} = \|\left\langle \sigma \right\rangle^b \widehat{F}\|_{L^2}$$
$$\text{where} \quad \widehat{F} = \widehat{F}(\tau, \xi_1, \xi_2)$$

The reason that they are useful is the following
:

(see D. Tataru)

for $0 < b < \dfrac{1}{2}$, $p, q > 2$

$\|F\|_{L^p(dt)L^q(dx_1)L^2(dx_2)} \leq_{C_b} \|F\|_{X^b}$

and

$\|F\|_{X^{-b}} \leq_{C_b} \|F\|_{L^{p'}(dt)L^{q'}(dx_1)L^2(dx_2)}$

where $\dfrac{2}{p} + \dfrac{3}{q} = \dfrac{5-2b}{2}$

We see that we can get arbitrarily close to the Strichartz exponents by choosing b close to $1/2$. The estimates (however) are not valid for $\beta = 1/2$.

For $\beta < 1$ we can work with $D^{\frac{1}{2}+}$ derivatives and balance the inequalities. However, in order to work with Bourgain spaces we need to localize in time on some fixed interval $[0, T]$ which implies that all our constants in the inequalities will depend on T. Thus we obtain a local in time result on some fixed interval $[0, T]$. This is not desirable and we need to put on the right hand side of our inequalities dual Strichartz (instead of Bourgain spaces) with constant independent of time.

In addition (as we saw earlier) it is advantageous to use Strichartz in rotated coordinates since for some terms it is useful to use x_{1+2}, x_{1-2} coordinates while on others it is better to use x_1, x_2. Here is a linear estimate regarding this which is new and it has its own interest. The proof was inspired by Journe Soffer and Sogge as well as Frank, Lewin, Lieb and Seiringer. Consider

$$\left\{\frac{1}{i}\partial_t - \Delta_1 - \Delta_2\right\}\Lambda = F \quad , \quad \Lambda(0) = 0$$

then

$$\left\|\Lambda\right\|_{L^2(dt)L^6(dx_1)L^2(dx_2)} \leq C$$
$$\left\|F\right\|_{L^2(dt)L^{6/5}(dx_{1-2})L^2(dx_{1+2})}$$

Since this is an end-point estimate one has to use the method (or idea) of Keel and Tao which however works in the same way. The key ingredient is an estimate on the Green's function.

We need to show that collapsing estimates work when one puts the right hand side in dual Strichartz. Consider (again)

$$\left\{\frac{1}{i}\partial_t - \Delta_1 - \Delta_2\right\}\Lambda = F \quad , \quad \Lambda(0) = 0$$

We want an estimate of the form,

$$\left\|(|\partial_t|^{\frac{1}{4}} + D_{1+2}^{\frac{1}{2}})\Lambda(t, x_{1+2}, x_{1-2})\right\|_{L_{\text{coll}}}$$
$$\leq_C \left\|D_1^{\frac{1}{2}} F D_2^{\frac{1}{2}}\right\|_{\widetilde{S}'}$$

where

$$L_{\text{coll}} := L^{\infty}(dx_{1-2})L^2(dtdx_{1+2})$$
and $\widetilde{S}'$ = dual Strichartz with $p' > 2$

The fact that end-point is excluded is due the the Christ-Kiselev Lemma.

• Xiaoqi Huang figured out how to treat the critical ($\beta = 1$) case (assuming small potential) . This involves some novel estimates.

For the previous estimate the time derivative is harder to handle. For the space derivative we can use the Christ Kiselev lemma and duality. Let me explain.

$$\left|\left\langle e^{it(\Delta_1+\Delta_2)} f(x_1,x_2), G(t,x_1,x_2)\right\rangle_{dtdx_1dx_2}\right|$$
$$= \left|\left\langle f, \widetilde{G}(\Delta_1+\Delta_2,\cdot)\right\rangle\right| = \left|\left\langle \widehat{f}, \widehat{G}(|\xi_1|^2+|\xi_2|^2,\cdot)\right\rangle\right|$$
$$\leq_C \|f\|_{L^2}\|G\|_{\mathcal{S}'}$$

therefore

$$\|\widehat{G}(|\xi_1|^2+|\xi_2|^2,\xi_1,\xi_2)\|_{L^2} \leq \|G\|_{\mathcal{S}'}$$

this is a Fourier restriction estimate

$\mathcal{S}' =$ dual Strichartz

We also know for $f(x_1,x_2)$

$$\left\|D^{1/2}_{1+2} e^{it(\Delta_1+\Delta_2)} f\right\|_{L_{\text{coll}}} \leq_C \left\|D_1^{1/2} f D_2^{1/2}\right\|_{L^2}$$

where $(F(s,x_1,x_2))$

$$\widetilde{F}(\Delta_1+\Delta_2,\cdot) := \int_{-\infty}^{+\infty} e^{-is(\Delta_1+\Delta_2)} F(s,\cdot)ds$$

Now let us consider,

$$D_{1+2}^{1/2} \int_{-\infty}^{+\infty} e^{i(t-s)(\Delta_1+\Delta_2)} F(s,\cdot)ds$$
$$= \int_{-\infty}^{+\infty} D_{1+2}^{1/2} e^{i(t-s)(\Delta_1+\Delta_2)} F(s,\cdot)ds$$
$$= D_{1+2}^{1/2} e^{it(\Delta_1+\Delta_2)} \widetilde{F}(\Delta_1+\Delta_2,\cdot)$$

and

$$\left\| D_{1+2}^{1/2} e^{it(\Delta_1+\Delta_2)} \widetilde{F}(\Delta_1+\Delta_2,\cdot) \right\|_{L_{\mathrm{coll}}}$$
$$\leq_C \left\| D_1^{1/2} \widetilde{F}(\Delta_1+\Delta_2,\cdot) D_2^{1/2} \right\|_{L^2}$$
$$\leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{\mathcal{S}'}$$

where we combined the previous collapsing estimate and the Fourier restriction theorem.

Now Christ Kiselev implies,

$$\left\| \int_0^t D_{1+2}^{1/2} e^{i(t-s)(\Delta_1+\Delta_2)} F(s,\cdot) ds \right\|_{L_{\text{coll}}}$$
$$\leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{\widetilde{S}'}$$

excluding the end point. Because the Christ Kiselev Lemma requires that if the time integration on the left is in L^2 the integration on the right must be in $L^{p'}$ for some $p' < 2$.

Time derivatives

Regarding time derivatives, we cannot use the Christ Kiselev lemma directly. So we need something new. Define the function ψ,

$$\psi(t) := \int_0^t e^{i(t-s)(\Delta_1+\Delta_2)} F(s,\cdot) ds$$

For simplicity we will write

$$\mathbf{\Delta} := \Delta_1 + \Delta_2$$

Now consider the time difference with step h,

$$\delta_{1/4}[h]\psi(t) := \frac{\psi(t+h) - \psi(t)}{h^{1/4}}$$

Since the difference captures the derivative at frequencies $1/h$ we will estimate the sum

$$\sum_j \left\| \delta_{1/4}[h_j]\psi \right\|_{L_{\text{coll}}} \leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{\widetilde{S}'}$$

where $h_j := 2^{-j}$

Now we compute,

$$\delta_{1/4}[h]\psi = \frac{1}{h^{1/4}}\int_0^h e^{is\Delta}F(t+s,\cdot)ds$$
$$+\frac{e^{ih\Delta}-1}{h^{1/4}}e^{it\Delta}\int_0^t e^{is\Delta}F(s,\cdot)ds$$
$$:= R_h[F]+K_h[F]$$

The second integral K_h is easier to estimate using Christ Kiselev and we leave it for later. For the first integral we will do a direct computation. We denote,

$$\Xi := (\xi_1,\xi_2)$$

and compute the Fourier transform

$$\mathcal{F}_{x_1,x_2\mapsto\xi_1,\xi_2}\{R_h[F]\}$$
$$= \int_{\mathbb{R}} d\sigma \left\{\widehat{\chi}_h(\sigma)e^{it(|\Xi|^2-\sigma)}\widehat{F}\Big(|\Xi|^2-\sigma,\Xi\Big)\right\}$$
$$\text{where } : \chi_h(s) := \frac{\chi_{[0,t]}(s)}{h^{1/4}}$$
$$\text{with } \chi_{[0,h]} := \text{characteristic function}$$

Now let us look at the integral,

$$Q_h := \int_{x_{1-2}=0} dt dx_{1+2} \left\{ \overline{R_h[F]} R_h[F] \right\}$$

With the frequency variables

$$\Xi := (\xi_1, \xi_2) \quad , \quad \Xi' := (\xi_3, \xi_4)$$

the collapse along the diagonal and integration over time produce the collapsing mechanism described by

$$\mathcal{C} := \delta(\xi_{1+2} - \xi_{3+4}) \delta\Big(|\xi_{1-2}|^2 - \sigma_1 - |\xi_{3-4}|^2 + \sigma_2 \Big)$$

In view of this let us define

$$\begin{aligned} &\xi := \xi_{1+2} = \xi_{3+4} \\ &E_1 := |\xi_{1-2}|^2 \quad , \quad E_2 := |\xi_{3-4}|^2 \\ &\tau := |\xi|^2 + E_1 - \sigma_1 = |\xi|^2 - E_2 - \sigma_2 \end{aligned}$$

So now the collapsing mechanism is described by

$$\mathcal{C} := \delta(\xi_{1+2} - \xi_{3+4})\delta\Big(E_1 - E_2 + \sigma_2 - \sigma_1\Big)$$

The integral Q_h becomes

$$Q_h = \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^4} d\sigma_1 d\sigma_2 dE_1 dE_2 (E_1 E_2)^{1/2} \times$$
$$\underline{\delta(E_1 - E_2 - \sigma_1 + \sigma_2)}\widehat{\chi}_h(\sigma_1)\widehat{\chi}_h(\sigma_2) \times$$
$$\widehat{F}_{\mathsf{av}}(\tau, \xi, E_1)\widehat{F}_{\mathsf{av}}(t, \xi, E_2)$$

where

$$\widehat{F}_{\mathsf{av}}(t, \xi, E) := \int\limits_{|\omega|=1} \widehat{F}(t, \xi + \sqrt{E}\omega, \xi - \sqrt{E}\omega) d\omega$$

is the angular average

Notice that we have the estimate,

$$|\widehat{F}_{\mathsf{av}}| \leq \frac{1}{\sqrt{|\xi| + \sqrt{E}}} \times$$
$$\sqrt{\int\limits_{|\omega|=1} |\xi - \sqrt{E}\omega| |\widehat{F}(t, \xi + \sqrt{E}\omega, \xi - \sqrt{E}\omega)|^2 d\omega}$$

and define for simplicity,

$$A_{\widehat{F}}(t,\xi,E) :=$$

$$:= \sqrt{\int_{|\omega|=1} |\xi - \sqrt{E}\omega||\widehat{F}(t,\xi+\sqrt{E}\omega,\xi-\sqrt{E}\omega)|^2 d\omega}$$

We rotate coordinates as follows

$$\frac{1}{2}\Big(E_1+E_2+\sigma_1+\sigma_2\Big) := \widetilde{E}_1$$
$$\frac{1}{2}\Big(E_1-E_2+\sigma_1-\sigma_2\Big) := \widetilde{E}_2$$
$$\frac{1}{2}\Big(E_1+E_2-\sigma_1-\sigma_2\Big) := \tilde{\sigma}_1$$
$$\frac{1}{2}\Big(E_1-E_2-\sigma_1+\sigma_2\Big) := \tilde{\sigma}_2$$

and

collapsing mechanism is $\delta(\tilde{\sigma}_2)$
while notice that $\tilde{\sigma}_1 = \tau - |\xi|^2$

We can solve for the original variables

$$E_1 = \frac{1}{2}\Big(\widetilde{E}_1 + \widetilde{E}_2 + \tau - |\xi|^2\Big)$$
$$E_2 = \frac{1}{2}\Big(\widetilde{E}_1 - \widetilde{E}_2 + \tau - |\xi|^2\Big)$$
$$\sigma_1 = \frac{1}{2}\Big(\widetilde{E}_1 + \widetilde{E}_2 - \tau + |\xi|^2\Big)$$
$$\sigma_2 = \frac{1}{2}\Big(\widetilde{E}_1 - \widetilde{E}_2 - \tau + |\xi|^2\Big)$$

For convenience let us also write

$$\widehat{E}_1 := \frac{1}{2}\Big(\widetilde{E}_1 + \widetilde{E}_2\Big)$$
$$\widehat{E}_2 := \frac{1}{2}\Big(\widetilde{E}_1 - \widetilde{E}_2\Big)$$
$$q := \frac{1}{2}\Big(\tau - |\xi|^2\Big)$$

and we have

$$Q_h \leq_C \int_{\mathbb{R}\times\mathbb{R}^3} d\tau d\xi \int_{\mathbb{R}^2} d\widehat{E}_1 d\widehat{E}_2$$
$$\times|\widehat{\chi}_h(\widehat{E}_1 - q)|\widehat{E}_1 - q|^{1/4}\frac{|\widehat{E}_1 + q|^{1/4} A_{\widehat{F}}(\tau,\xi,\widehat{E}_1 + q)}{|\widehat{E}_1 - q|^{1/4}}$$
$$\times|\widehat{\chi}_h(\widehat{E}_2 - q)||\widehat{E}_2 - q|^{1/4}\frac{|\widehat{E}_2 + q|^{1/4} A_{\widehat{F}}(\tau,\xi,\widehat{E}_2 + q)}{|\widehat{E}_2 - q|^{1/4}}$$

Let us look at one of the integrals, let's say for $\widehat{E}_1$ which we now write simply as E can be estimated as follows,

$$\int dE|\widehat{\chi}_h(E-q)||E-q|^{1/4}\frac{|E+q|^{1/4} A_{\widehat{F}}(\tau,\xi,E+q)}{|E-q|^{1/4}}$$
$$\leq \left(\int d\sigma|\widehat{\chi}_h(\sigma)|^2|\sigma|^{1/2}\right)^{1/2}$$
$$\times\left(\int dE\frac{|E+q|^{1/2} A^2_{\widehat{F}}(t,\xi,E+q)}{|E-q|^{1/2}}\right)^{1/2}$$

We will look later at the integral

$$\int d\sigma|\widehat{\chi}_h(\sigma)|^2|\sigma|^{1/2}$$

Consider now the integral,

$$I := \int dE \frac{|E+q|^{1/2} A^2_{\widehat{F}}(t,\xi,E+q)}{|E-q|^{1/2}}$$

We can rewrite it by defining

$$E+q := \rho^2 \quad , \quad \xi_{1-2} := \rho\omega$$

so we have $E - q = |\xi|^2 + \rho^2 - \tau$

hence

$$I = \int_{\mathbb{R}^3} d\xi_{1-2} \frac{|\xi_2||\widehat{F}(\tau,\xi_1,\xi_2)|^2}{\Big||\xi_{1-2}|^2 + |\xi_{1-2}|^2 - \tau\Big|^{1/2}}$$

Using this information

we have the estimate

$$Q_h \leq_C \int d\tau d\xi_1 d\xi_2 \frac{|\xi_2||\widehat{F}(\tau,\xi_1,\xi_2)|^2}{\Big||\xi_1|^2 + |\xi_2|^2 - \tau\Big|^{1/2}}$$
$$\leq_C \left\| F D_2^{1/2} \right\|_{L^2(dt)L^{3/2}(dx_1)L^2(dx_2)}$$
$$\leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{L^2(dt)L^{6/5}(dx_1)L^2(dx_2)}$$

where in the first line we used the Bourgain spaces estimate and in the second Sobolev. Now let us consider $h_j = 2^j$ and decompose the function $\widehat{F}$ by projecting onto parabolic regions such that

$$|\sigma| := \tau - |\xi_1|^2 - |\xi_2|^2 \sim 2^k$$
$$\text{while } h_j := 2^{-j}$$

Let us call the projection onto parabolic regions

$$P_k^{\text{par}} := \text{projection on range } \ |\sigma| \sim 2^k$$

The point of the projection is that

$$\begin{aligned}
&\int_{\sigma\sim 2^k} |\widehat{\chi}_{h_j}(\sigma)||\sigma|^{1/2} d\sigma \\
&= \int_{\sigma\sim 2^k} \frac{|e^{ih_j\sigma}-1|^2}{|h_j\sigma|^{1/2}} \frac{d\sigma}{\sigma} \\
&\le_C \min\left\{(2^{-j}2^k)^{3/2}, (2^{-j}2^k)^{-1/2}\right\}
\end{aligned}$$

This is small in the off diagonal cases $j << k$ or $j >> k$. Summing over j, k gives the desired estimate.

The term K_h is easier to handle. Standard methods imply

$$\left\|\frac{e^{h\Delta}-1}{h^{1/4}}e^{it\Delta}f\right\|_{L_{\text{coll}}} \leq_C \left\|D_1^{1/2} f D_2^{1/2}\right\|_{L^2}$$

and with P_k denoting the projection onto frequencies

$$\sqrt{|\xi_1|^2+|\xi_2|^2} \sim 2^k$$

and $h_j = 2^{-j}$ we have a frequency localized estimate

$$\left\|\frac{e^{h_j\Delta}-1}{h^{1/4}}e^{it\Delta}P_k f\right\|_{L_{\text{coll}}} \leq_C$$
$$\min\left\{(2^{-j}2^{2k})^{3/4},(2^{-j}2^{2k})^{-1/4}\right\}$$
$$\times\left\|D_1^{1/2}P_k f D_2^{1/2}\right\|_{L^2}$$

Now,

$$\left\| \frac{e^{ih\Delta} - 1}{h^{1/4}} e^{it\Delta} \int_{-\infty}^{+\infty} e^{-is\Delta} F(s, \cdot) ds \right\|_{L_{\text{coll}}}$$
$$\leq_C \left\| D_1^{1/2} \widetilde{F}(\Delta, \cdot) D_2^{1/2} \right\|_{L^2}$$
$$\leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{\mathcal{S}'}$$

from the Fourier restriction theorem on the parabola. Christ Kiselev lemma implies the same estimate with the end point excluded.

$$\left\| \frac{e^{ih\Delta} - 1}{h^{1/4}} e^{it\Delta} \int_{0}^{t} e^{-is\Delta} F(s, \cdot) ds \right\|_{L_{\text{coll}}}$$
$$\leq_C \left\| D_1^{1/2} F D_2^{1/2} \right\|_{\widetilde{\mathcal{S}}'}$$

Finally if we pick $h_j = 2^{-j}$ and project using P_k. We have off diagonal decay etc.